1. Two exercises on $\sin k^\circ \sin (k + 1)^\circ$:

(a) [AIME2 2000] Find the smallest positive integer $n$ such that
\[
\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.
\]

(b) Prove that
\[
\frac{1}{\sin 1^\circ \sin 2^\circ} + \frac{1}{\sin 2^\circ \sin 3^\circ} + \cdots + \frac{1}{\sin 89^\circ \sin 90^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.
\]

**Solution:** Note that
\[
\sin 1^\circ = \sin[(x + 1)^\circ - x^\circ] = \sin(x + 1)^\circ \cos x^\circ - \cos(x + 1)^\circ \sin x^\circ.
\]
Thus
\[
\frac{\sin 1^\circ}{\sin x^\circ \sin(x + 1)^\circ} = \frac{\cos x^\circ \sin(x + 1)^\circ - \sin x^\circ \cos(x + 1)^\circ}{\sin x^\circ \sin(x + 1)^\circ} = \cot x^\circ - \cot(x + 1)^\circ.
\]
(a) Multiplying both sides of the given equation by \( \sin 1^\circ \), we have
\[
\frac{\sin 1^\circ}{\sin n^\circ} = (\cot 45^\circ - \cot 46^\circ) + (\cot 47^\circ - \cot 48^\circ)
+ \cdots + (\cot 133^\circ - \cot 134^\circ)
= \cot 45^\circ - (\cot 46^\circ + \cot 134^\circ) + (\cot 47^\circ + \cot 133^\circ)
- \cdots + (\cot 89^\circ + \cot 91^\circ) - \cot 90^\circ
= 1.
\]
Therefore, \( \sin n^\circ = \sin 1^\circ \), and the least possible integer value for \( n \) is 1.

(b) The left-hand side of the desired equation is equal to
\[
\sum_{k=1}^{89} \frac{1}{\sin k^\circ \sin (k+1)^\circ} = \frac{1}{\sin 1^\circ} \sum_{k=1}^{89} [\cot k^\circ - \cot (k+1)^\circ]
= 1 \cdot \cot 1^\circ = \frac{\cos 1^\circ}{\sin^2 1^\circ},
\]
thus completing the proof.

2. [China 2001, by Xiaoyang Su] Let \( \triangle ABC \) be a triangle, and let \( x \) be a nonnegative real number. Prove that
\[
a^x \cos A + b^x \cos B + c^x \cos C \leq \frac{1}{2} (a^x + b^x + c^x).
\]

**Solution:** By symmetry, we may assume that \( a \geq b \geq c \). Hence \( A \geq B \geq C \), and so \( \cos A \leq \cos B \leq \cos C \). Thus
\[
(a^x - b^x)(\cos A - \cos B) \leq 0,
\]
or
\[
a^x \cos A + b^x \cos B \leq a^x \cos B + b^x \cos A.
\]
Adding the last inequality with its analogous cyclic symmetric forms and then adding \( a^x \cos A + b^x \cos B + c^x \cos C \) to both sides of the resulting inequality gives
\[
3(a^x \cos A + b^x \cos B + c^x \cos C)
\leq (a^x + b^x + c^x)(\cos A + \cos B + \cos C),
\]
from which the desired result follows as a consequence of Introductory Problem 27(b).

**Note:** The above solution is similar to the proof of **Chebyshev’s inequality**. We can also apply the **rearrangement inequality** to simplify our work. Because \( a \geq b \geq c \) and \( \cos A \leq \cos B \leq \cos C \), we have

\[
a^x \cos A + b^x \cos B + c^x \cos C \leq a^x \cos B + b^x \cos C + c^x \cos A
\]

and

\[
a^x \cos A + b^x \cos B + c^x \cos C \leq a^x \cos C + b^x \cos A + c^x \cos B.
\]

Hence

\[
3(a^x \cos A + b^x \cos B + c^x \cos C) \\
\leq (a^x + b^x + c^x)(\cos A + \cos B + \cos C).
\]

3. Let \( x, y, z \) be positive real numbers.

(a) Prove that

\[
\frac{x}{\sqrt{1+x^2}} + \frac{y}{\sqrt{1+y^2}} + \frac{z}{\sqrt{1+z^2}} \leq \frac{3\sqrt{3}}{2}
\]

if \( x + y + z = xyz \);

(b) Prove that

\[
\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} \geq \frac{3\sqrt{3}}{2}
\]

if \( 0 < x, y, z < 1 \) and \( xy + yz + zx = 1 \).

**Solution:** Both problems can be solved by trigonometric substitutions.

(a) By Introductory Problem 20(a), there is an acute triangle \( ABC \) with \( \tan A = x, \tan B = y, \) and \( \tan C = z \). Note that

\[
\frac{\tan A}{\sqrt{1 + \tan^2 A}} = \frac{\tan A}{\sec A} = \sin A.
\]

The desired inequality becomes

\[
\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2},
\]

which is Introductory Problem 28(c).
(b) From the given condition and Introductory Problem 19(a), we can assume that there is an acute triangle $ABC$ such that

$$\tan \frac{A}{2} = x, \quad \tan \frac{B}{2} = y, \quad \tan \frac{C}{2} = z.$$  

By the **double-angle formulas**, it suffices to prove that

$$\tan A + \tan B + \tan C \geq 3\sqrt{3},$$

which is Introductory Problem 20(b).

4. [China 1997] Let $x, y, z$ be real numbers with $x \geq y \geq z \geq \pi/12$ such that $x + y + z = \pi/2$. Find the maximum and the minimum values of the product $\cos x \sin y \cos z$.

**Solution:** Let $p = \cos x \sin y \cos z$. Because $\pi/2 \geq y \geq z$, $\sin(y - z) \geq 0$. By the **product-to-sum formulas**, we have

$$p = \frac{1}{2} \cos x [\sin(y + z) + \sin(y - z)] \geq \frac{1}{2} \cos x \sin(y + z) = \frac{1}{2} \cos^2 x.$$  

Note that $x = \pi/2 - (y + z) \leq \pi/2 - 2 \cdot \pi/12 = \pi/3$. Hence the minimum value of $p$ is $\frac{1}{2} \cos^2 \pi/3 = \frac{1}{8}$, obtained when $x = \pi/3$ and $y = z = \pi/12$.

On the other hand, we also have

$$p = \frac{1}{2} \cos z [\sin(x + y) - \sin(x - y)] \leq \frac{1}{2} \cos^2 z,$$

by noting that $\sin(x - y) \geq 0$ and $\sin(x + y) = \cos z$. By the **double-angle formulas**, we deduce that

$$p \leq \frac{1}{4} (1 + \cos 2z) \leq \frac{1}{4} \left(1 + \cos \frac{\pi}{6}\right) = \frac{2 + \sqrt{3}}{8}.$$

This maximum value is obtained if and only if $x = y = \frac{5\pi}{24}$ and $z = \frac{\pi}{12}$.

5. Let $ABC$ be an acute-angled triangle, and for $n = 1, 2, 3$, let

$$x_n = 2^{n-3}(\cos^n A + \cos^n B + \cos^n C) + \cos A \cos B \cos C.$$  

Prove that

$$x_1 + x_2 + x_3 \geq \frac{3}{2}.$$
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Solution: By the arithmetic–geometric means inequality,

\[ \cos^3 x + \frac{\cos x}{4} \geq \cos^2 x \]

for \(x\) such that \(\cos x \geq 0\). Because triangle \(ABC\) is acute, \(\cos A\), \(\cos B\), and \(\cos C\) are nonnegative. Setting \(x = A\), \(x = B\), \(x = C\) and adding the resulting inequalities yields

\[ x_1 + x_3 \geq \cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 2x_2. \]

Consequently,

\[ x_1 + x_2 + x_3 \geq 3x_2 = \frac{3}{2}, \]

by Introductory Problem 24(d).

6. Find the sum of all \(x\) in the interval \([0, 2\pi]\) such that

\[ 3 \cot^2 x + 8 \cot x + 3 = 0. \]

Solution: Consider the quadratic equation

\[ 3u^2 + 8u + 3 = 0. \]

The roots of the above equation are \(u_1 = \frac{-8 + 2\sqrt{7}}{6}\) and \(u_2 = \frac{-8 - 2\sqrt{7}}{6}\). Both roots are real, and their product \(u_1u_2\) is equal to \(-1\) (by Viète’s theorem).

Because \(y = \cot x\) is a bijection from the interval \((0, \pi)\) to the real numbers, there is a unique pair of numbers \(x_{1,1}\) and \(x_{2,1}\) with \(0 < x_{1,1}, x_{2,1} < \pi\) such that \(\cot x_{1,1} = u_1\) and \(\cot x_{2,1} = u_2\). Because \(u_1, u_2\) are negative, \(\frac{\pi}{2} < x_{1,1}, x_{2,1} < \pi\), and so \(\pi < x_{1,1} + x_{2,1} < 2\pi\). Because \(\cot x \tan x = 1\) and both \(\tan x\) and \(\cot x\) have period \(\pi\), it follows that

\[ 1 = \cot x \tan x = \cot x \cot \left( \frac{\pi}{2} - x \right) = \cot x \cot \left( \frac{3\pi}{2} - x \right) \]

\[ = \cot x_{1,1} \cot x_{2,1}. \]

Therefore, \(x_{1,1} + x_{2,1} = \frac{3\pi}{2}\). Likewise, in the interval \((\pi, 2\pi)\), there is a unique pair of numbers \(x_{1,2}\) and \(x_{2,2}\) satisfying the conditions of the problem with \(x_{1,2} + x_{2,2} = \frac{7\pi}{2}\). Thus the answer to the problem is \(x_{1,1} + x_{2,1} + x_{1,2} + x_{2,2} = 5\pi\).
7. Let $ABC$ be an acute-angled triangle with area $K$. Prove that

$$\sqrt{a^2b^2 - 4K^2} + \sqrt{b^2c^2 - 4K^2} + \sqrt{c^2a^2 - 4K^2} = \frac{a^2 + b^2 + c^2}{2}.$$ 

**Solution:** We have $2K = ab \sin C = bc \sin A = ca \sin B$. The expression on the left-hand side of the desired equation is equal to

$$\sqrt{a^2b^2 - a^2b^2 \sin^2 C} + \sqrt{b^2c^2 - b^2c^2 \sin^2 A} + \sqrt{c^2a^2 - c^2a^2 \sin^2 B}$$

$$= ab \cos C + bc \cos A + ca \cos B$$

$$= \frac{a}{2}(b \cos C + c \cos B) + \frac{b}{2}(c \cos A + a \cos C)$$

$$+ \frac{c}{2}(a \cos B + b \cos A)$$

$$= \frac{a}{2} \cdot a + \frac{b}{2} \cdot b + \frac{c}{2} \cdot c,$$

and the conclusion follows.

**Note:** We encourage the reader to explain why this problem is the equality case of Advanced Problem 42(a).

8. Compute the sums

$$\binom{n}{1} \sin a + \binom{n}{2} \sin 2a + \cdots + \binom{n}{n} \sin na$$

and

$$\binom{n}{1} \cos a + \binom{n}{2} \cos 2a + \cdots + \binom{n}{n} \cos na.$$ 

**Solution:** Let $S_n$ and $T_n$ denote the first and second sums, respectively. Set the complex number $z = \cos a + i \sin a$. Then, by de Moivre’s formula, we have $z^n = \cos na + i \sin na$. By the binomial theorem, we obtain

$$1 + T_n + iS_n = 1 + \binom{n}{1}(\cos a + i \sin a) + \binom{n}{2}(\cos 2a + i \sin 2a)$$

$$+ \cdots + \binom{n}{n}(\cos na + i \sin na)$$

$$= \binom{n}{0}z^0 + \binom{n}{1}z + \binom{n}{2}z^2 + \cdots + \binom{n}{n}z^n$$

$$= (1 + z)^n.$$
Because
\[ 1 + z = 1 + \cos a + i \sin a = 2 \cos^2 \frac{a}{2} + 2i \sin \frac{a}{2} \cos \frac{a}{2} \]
\[= 2 \cos \frac{a}{2} \left( \cos \frac{a}{2} + i \sin \frac{a}{2} \right), \]
it follows that
\[(1 + z)^n = 2^n \cos^n \frac{a}{2} \left( \cos \frac{na}{2} + i \sin \frac{na}{2} \right), \]
again by de Moivre’s formula. Therefore,
\[(1 + T_n) + iS_n = \left(2^n \cos^n \frac{a}{2} \cos \frac{na}{2}\right) + i \left(2^n \cos^n \frac{a}{2} \sin \frac{na}{2}\right), \]
and so
\[S_n = 2^n \cos^n \frac{a}{2} \sin \frac{na}{2} \quad \text{and} \quad T_n = -1 + 2^n \cos^n \frac{a}{2} \cos \frac{na}{2}. \]

9. [Putnam 2003] Find the minimum value of
\[| \sin x + \cos x + \tan x + \cot x + \sec x + \csc x|\]
for real numbers \(x\).

**Solution:** Set \(a = \sin x\) and \(b = \cos x\). We want to minimize
\[P = \left| a + b + \frac{a}{b} + \frac{b}{a} + \frac{1}{a} + \frac{1}{b} \right|\]
\[= \left| \frac{ab(a + b) + a^2 + b^2 + a + b}{ab} \right| \]
Note that \(a^2 + b^2 = \sin^2 x + \cos^2 x = 1\). Set \(c = a + b\). Then \(c^2 = (a + b)^2 = 1 + 2ab\), and so \(2ab = c^2 - 1\). Note also that by the addition and subtraction formulas, we have
\[c = \sin x + \cos x = \sqrt{2} \left( \frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x \right) = \sqrt{2} \sin \left( \frac{\pi}{4} + x \right), \]
and so the range of \( c \) is the interval \([-\sqrt{2}, \sqrt{2}]\). Consequently, it suffices to find the minimum of

\[
P(c) = \left| \frac{2ab(a + b) + 2 + 2(a + b)}{2ab} \right| = \left| \frac{c(c^2 - 1) + 2(c + 1)}{c^2 - 1} \right| = \left| c + \frac{2}{c - 1} \right| = \left| c - 1 + \frac{2}{c - 1} + 1 \right|.
\]

for \( c \) in the interval \([-\sqrt{2}, \sqrt{2}]\). If \( c - 1 > 0 \), then by the arithmetic–geometric means inequality, \((c - 1) + \frac{2}{c - 1} > 2\sqrt{2}\), and so \( P(c) > 1 + 2\sqrt{2} \). If \( c - 1 < 0 \), then by the same token,

\[
(c - 1) + \frac{2}{c - 1} = -\left( (1 - c) + \frac{2}{1 - c} \right) \leq -2\sqrt{2},
\]

with equality if and only if \( 1 - c = \frac{2}{1 - c} \), or \( c = 1 - \sqrt{2} \). It follows that the minimum value sought is \( -2\sqrt{2} + 1 = 2\sqrt{2} - 1 \), obtained when \( c = 1 - \sqrt{2} \).

**Note:** Taking the derivative of the function

\[
f(x) = \sin x + \cos x + \tan x + \cot x + \sec x + \csc x
\]

and considering only its critical points is a troublesome approach to this problem, because it is difficult to show that \( f(x) \) does not cross the \( x \) axis smoothly. Indeed, with a little bit more work, we can show that \( f(x) \neq 0 \) with the presented solution.

10. [Belarus 1999] Two real sequences \( x_1, x_2, \ldots \) and \( y_1, y_2, \ldots \) are defined in the following way:

\[
x_1 = y_1 = \sqrt{3}, \quad x_{n+1} = x_n + \sqrt{1 + x_n^2}, \quad y_{n+1} = \frac{y_n}{1 + \sqrt{1 + y_n^2}},
\]

for all \( n \geq 1 \). Prove that \( 2 < x_n y_n < 3 \) for all \( n > 1 \).

**Solution:** Writing \( x_n = \tan a_n \) for \( 0^\circ < a_n < 90^\circ \), by the half-angle formula we have

\[
x_{n+1} = \tan a_n + \sqrt{1 + \tan^2 a_n} = \tan a_n + \sec a_n = \frac{1 + \sin a_n}{\cos a_n} = \tan \left( \frac{90^\circ + a_n}{2} \right).
\]
Because $a_1 = 60^\circ$, we have $a_2 = 75^\circ$, $a_3 = 82.5^\circ$, and in general $a_n = 90^\circ - \frac{30^\circ}{2n-1}$. Thus

$$x_n = \tan \left( 90^\circ - \frac{30^\circ}{2n-1} \right) = \cot \left( \frac{30^\circ}{2n-1} \right) = \cot \theta_n,$$

where $\theta_n = \frac{30^\circ}{2n-1}$.

A similar calculation shows that

$$y_n = \tan 2\theta_n = \frac{2 \tan \theta_n}{1 - \tan^2 \theta_n},$$

implying that

$$x_n y_n = \frac{2}{1 - \tan^2 \theta_n}.$$

Because $0^\circ < \theta_n < 45^\circ$, we have $0 < \tan^2 \theta_n < 1$ and $x_n y_n > 2$. For $n > 1$, we have $\theta_n < 30^\circ$, implying that $\tan^2 \theta_n < \frac{1}{3}$ and $x_n y_n < 3$.

11. Let $a$, $b$, $c$ be real numbers such that

$$\sin a + \sin b + \sin c \geq \frac{3}{2}.$$

Prove that

$$\sin \left( a - \frac{\pi}{6} \right) + \sin \left( b - \frac{\pi}{6} \right) + \sin \left( c - \frac{\pi}{6} \right) \geq 0.$$

**Solution:** Assume for contradiction that

$$\sin \left( a - \frac{\pi}{6} \right) + \sin \left( b - \frac{\pi}{6} \right) + \sin \left( c - \frac{\pi}{6} \right) < 0.$$

Then by the **addition and subtraction formulas**, we have

$$\frac{1}{2} (\cos a + \cos b + \cos c) > \frac{\sqrt{3}}{2} (\sin a + \sin b + \sin c) \geq \frac{3\sqrt{3}}{4}.$$

It follows that

$$\cos a + \cos b + \cos c > \frac{3\sqrt{3}}{2},$$
which implies that
\[
\sin \left( a + \frac{\pi}{3} \right) + \sin \left( b + \frac{\pi}{3} \right) + \sin \left( c + \frac{\pi}{3} \right) = \frac{1}{2} \sin a + \sin b + \sin c + \frac{\sqrt{3}}{2} \cos a + \cos b + \cos c \geq \frac{1}{2} \cdot 3 \cdot \frac{3\sqrt{3}}{2} = 3,
\]
which is impossible, because \( \sin x < 1 \).

12. Consider any four numbers in the interval \( \left[ \frac{\sqrt{2} - \sqrt{6}}{2}, \frac{\sqrt{2} + \sqrt{6}}{2} \right] \). Prove that there are two of them, say \( a \) and \( b \), such that
\[
|a\sqrt{4 - b^2} - b\sqrt{4 - a^2}| \leq 2.
\]

**Solution:** Dividing both sides of the inequality by 4 yields
\[
\left| \frac{a}{2} \sqrt{1 - \left( \frac{b}{2} \right)^2} - \frac{b}{2} \sqrt{1 - \left( \frac{a}{2} \right)^2} \right| \leq \frac{1}{2}.
\]
We substitute \( \frac{a}{2} = \sin x \) and \( \frac{b}{2} = \sin y \). The last inequality reduces to
\[
| \sin(x - y) | = | \sin x \cos y - \sin y \cos x | \leq \sin \frac{\pi}{6}.
\]
We want to find \( t_1 \) and \( t_2 \) such that
\[
\sin t_1 = \frac{\sqrt{2} - \sqrt{6}}{4} \quad \text{and} \quad \sin t_2 = \frac{\sqrt{2} + \sqrt{6}}{4}.
\]
By the double-angle formulas, we conclude that \( \cos 2t_1 = 1 - 2 \sin^2 t_1 = 1 - \frac{8 - 4\sqrt{3}}{8} = \frac{\sqrt{3}}{2} = \cos \left( \pm \frac{\pi}{6} \right) \) and \( \cos 2t_2 = -\frac{\sqrt{3}}{2} = \cos \frac{5\pi}{6} \). Because \( y = \sin x \) is a **one-to-one** and **onto** map between the intervals \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) and \( [-1, 1] \), it follows that \( t_1 = -\frac{\pi}{12} \) and \( t_2 = \frac{5\pi}{12} \).

We divide the interval \( \left[ -\frac{\pi}{12}, \frac{5\pi}{12} \right] \) into three disjoint intervals of length \( \frac{\pi}{6} \):
\[
I_1 = \left[ -\frac{\pi}{12}, \frac{\pi}{12} \right), I_2 = \left[ \frac{\pi}{12}, \frac{\pi}{4} \right), \text{ and } I_3 = \left[ \frac{\pi}{4}, \frac{5\pi}{12} \right].
\]
The function \( y = 2 \sin x \) takes the intervals \( I_1, I_2, I_3 \) injectively and surjectively to the intervals \( I_1' = \)
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\( \left[ \frac{\sqrt{2}}{2} - \sqrt{\frac{6}{2}}, 2 \sin \frac{\pi}{12} \right], \ I_2 = \left[ 2 \sin \frac{\pi}{12}, \sqrt{2} \right], \ I_3' = \left[ \sqrt{2}, \frac{\sqrt{6}+\sqrt{2}}{2} \right], \) respectively.

By the pigeonhole principle, one of the intervals \( I_1', I_2' \), or \( I_3' \) contains two of the four given numbers, say \( a \) and \( b \). It follows that one of the intervals \( I_1, I_2, \) or \( I_3 \) contains \( a = 2 \sin x \) and \( b = 2 \sin y \). Because the intervals \( I_1, I_2, \) and \( I_3 \) have equal lengths of \( \frac{\pi}{6} \), it follows that \( |x - y| \leq \frac{\pi}{6} \).

We have obtained the desired inequality (\( \ast \)).

13. Let \( a \) and \( b \) be real numbers in the interval \([0, \frac{\pi}{2}]\). Prove that

\[
\sin^6 a + 3 \sin^2 a \cos^2 b + \cos^6 b = 1
\]

if and only if \( a = b \).

Solution: The first equality can be rewritten as

\[
(sin^2 a)^3 + (cos^2 b)^3 + (-1)^3 - 3(sin^2 a)(cos^2 b)(-1) = 0. \quad (\ast)
\]

We will use the identity

\[
x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2].
\]

Let \( x = \sin^2 a, y = \cos^2 b, \) and \( z = -1 \). According to equation (\( \ast \)) we have \( x^3 + y^3 + z^3 - 3xyz = 0 \). Hence \( x + y + z = 0 \) or \( (x - y)^2 + (y - z)^2 + (z - x)^2 = 0 \). The latter would imply \( x = y = z, \) or \( \sin^2 a = \cos^2 b = -1, \) which is impossible. Thus \( x + y + z = 0 \), so that \( \sin^2 a + \cos^2 b - 1 = 0, \) or \( \sin^2 a = 1 - \cos^2 b. \) It follows that \( \sin^2 a = \sin^2 b, \) and taking into account that \( 0 \leq a, b \leq \frac{\pi}{2}, \) we obtain \( a = b. \)

Even though all the steps above are reversible, we will show explicitly that if \( a = b, \) then

\[
\sin^6 a + \cos^6 a + 3 \sin^2 a \cos^2 b = 1.
\]

Indeed, the expression on the left-hand side could be written as

\[
(sin^2 a + \cos^2 a)(\sin^4 a - \sin^2 a \cos^2 a + \cos^4 a) + 3 \sin^2 a \cos^2 a
\]

\[
= (\sin^2 a + \cos^2 a)^2 - 3 \sin^2 a \cos^2 a + 3 \sin^2 a \cos^2 a = 1.
\]

14. Let \( x, y, z \) be real numbers with \( 0 < x < y < z < \frac{\pi}{2}, \) Prove that

\[
\frac{\pi}{2} + 2 \sin x \cos y + 2 \sin y \cos z \geq \sin 2x + \sin 2y + \sin 2z.
\]
Solution: By the double-angle formulas, the above inequalities reduce to

\[ \frac{\pi}{2} > 2 \sin x (\cos x - \cos y) + 2 \sin y (\cos y - \cos z) + 2 \sin z \cos z, \]

or

\[ \frac{\pi}{4} > \sin x (\cos x - \cos y) + \sin y (\cos y - \cos z) + \sin z \cos z. \]

As shown in Figure 5.1, in the rectangular coordinate plane, we consider points

\[ O = (0, 0), \quad A = (\cos x, \sin x), \quad A_1 = (\cos x, 0), \quad B = (\cos y, \sin y), \quad B_1 = (\cos y, 0), \quad B_2 = (\cos y, \sin x), \quad C = (\cos z, \sin z), \quad C_1 = (\cos z, 0), \quad C_2 = (\cos z, \sin y), \]

and \( D = (0, \sin z). \) Points \( A, B, \) and \( C \) are in the first quadrant of the coordinate plane, and they lie on the unit circle in counterclockwise order.

Let \( D \) denote the region enclosed by the unit circle in the first quadrant (including the boundary). It is not difficult to see that quadrilaterals \( AA_1B_1B_2, \)
\( BB_1C_1C_2, \) and \( CC_1OD \) are nonoverlapping rectangles inside region \( D. \) It is also not difficult to see that \( [D] = \frac{\pi}{4}, [AA_1B_1B_2] = \sin x (\cos x - \cos y), \)
\( [BB_1C_1C_2] = \sin y (\cos y - \cos z), \) and \( [CC_1OD] = \sin z \cos z, \) from which our desired result follows.

15. For a triangle \( XYZ, \) let \( r_{XYZ} \) denote its inradius. Given that the convex pentagon \( ABCDE \) is inscribed in a circle, prove that if \( r_{ABC} = r_{AED} \) and \( r_{ABD} = r_{AEC}, \) then triangles \( ABC \) and \( AED \) are congruent.

Solution: Let \( R \) be the radius of the circle in which \( ABCDE \) is inscribed.
As shown in the proof of Problem 27(a), if \( ABC \) is a triangle with inradius \( r \) and circumradius \( R \), then

\[
1 + \frac{r}{R} = \cos A + \cos B + \cos C = \cos A - \cos(A + C) + \cos C.
\]

Let \( 2a \), \( 2b \), \( 2c \), \( 2d \), and \( 2e \) be the measures of arcs \( \widehat{AB} \), \( \widehat{BC} \), \( \widehat{CD} \), \( \widehat{DE} \), and \( \widehat{EA} \), respectively. Then \( a + b + c + d + e = 180^\circ \). Because \( r_{ABC} = r_{AED} \) and \( r_{ABD} = r_{AEC} \), we have

\[
\cos a - \cos(a + b) + \cos b = \cos d + \cos e - \cos(d + e) \quad (\ast)
\]

and

\[
\cos a + \cos(b + c) + \cos(d + e) = \cos e + \cos(c + d) + \cos(a + b).
\]

Subtracting the two equations, we obtain \( \cos b + \cos(c + d) = \cos d + \cos(b + c) \), or

\[
2 \cos \frac{b + c + d}{2} \cos \frac{b - c - d}{2} = 2 \cos \frac{b + c + d}{2} \cos \frac{d - b - c}{2}
\]

by \textbf{sum-to-product formulas}. It follows that

\[
\cos \frac{b - c - d}{2} = \cos \frac{d - b - c}{2},
\]

and so \( b = d \). Plugging this result into equation (\ast) yields

\[
\cos a - \cos(a + b) + \cos b = \cos b + \cos e - \cos(b + e),
\]
or \( \cos a + \cos(b + e) = \cos e + \cos(a + b) \). Applying the sum-to-product formulas again gives
\[
2 \cos \frac{a + b + e}{2} \cos \frac{a - b - e}{2} = 2 \cos \frac{a + b + e}{2} \cos \frac{e - a - b}{2},
\]
and so \( \cos \frac{a-b-e}{2} = \cos \frac{e-a-b}{2} \). It follows that \( a = e \). Because \( a = e \) and \( b = d \), triangles \( ABC \) and \( AED \) are congruent.

16. All the angles in triangle \( ABC \) are less then 120°. Prove that
\[
\frac{\cos A + \cos B - \cos C}{\sin A + \sin B - \sin C} > \frac{\sqrt{3}}{3}.
\]

Solution: Consider the triangle \( A_1B_1C_1 \), as shown in Figure 5.3, where \( \angle A_1 = 120° - \angle A \), \( \angle B_1 = 120° - \angle B \), and \( \angle C_1 = 120° - \angle C \). The given condition guarantees the existence of such a triangle.

![Diagram](image)

Figure 5.3.

Applying the triangle inequality in triangle \( A_1B_1C_1 \) gives \( B_1C_1 + C_1A_1 > A_1B_1 \); that is
\[
\sin A_1 + \sin B_1 > \sin C_1
\]
by applying the law of sines to triangle \( A_1B_1C_1 \). It follows that
\[
\sin(120° - A) + \sin(120° - B) > \sin(120° - C),
\]
or
\[
\frac{\sqrt{3}}{2} (\cos A + \cos B - \cos C) + \frac{1}{2} (\sin A + \sin B - \sin C) > 0.
\]
Taking into account that \( a + b > c \) implies \( \sin A + \sin B - \sin C > 0 \), the above inequality can be rewritten as
\[
\frac{\sqrt{3}}{2} \cdot \frac{\cos A + \cos B - \cos C}{\sin A + \sin B - \sin C} + \frac{1}{2} > 0,
\]
from which the conclusion follows.
17. [USAMO 2002] Let $ABC$ be a triangle such that

$$\left(\cot \frac{A}{2}\right)^2 + \left(2\cot \frac{B}{2}\right)^2 + \left(3\cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where $s$ and $r$ denote its semiperimeter and its inradius, respectively. Prove that triangle $ABC$ is similar to a triangle $T$ whose side lengths are all positive integers with no common divisor and determine these integers.

**Solution:** Define

$$u = \cot \frac{A}{2}, \quad v = \cot \frac{B}{2}, \quad w = \cot \frac{C}{2}.$$

As shown in Figure 5.4, denote the incenter of triangle $ABC$ by $I$, and let $D$, $E$, and $F$ be the points of tangency of the incircle with sides $BC$, $CA$, and $AB$, respectively. Then $|EI| = r$, and by the standard formula, $|AE| = s - a$.

![Figure 5.4.](image)

We have

$$u = \cot \frac{A}{2} = \frac{|AE|}{|EI|} = \frac{s - a}{r},$$

and similarly $v = \frac{s - b}{r}$, $w = \frac{s - c}{r}$. Because

$$\frac{s}{r} = \frac{(s - a) + (s - b) + (s - c)}{r} = u + v + w,$$

we can rewrite the given relation as

$$49\left[u^2 + 4v^2 + 9w^2\right] = 36(u + v + w)^2.$$
Expanding the last equality and canceling like terms, we obtain
\[ 13u^2 + 160v^2 + 405w^2 - 72(uv + vw + wu) = 0, \]
or
\[ (3u - 12v)^2 + (4v - 9w)^2 + (18w - 2u)^2 = 0. \]
Therefore, \( u : v : w = 1 : \frac{1}{4} : \frac{1}{9} \). This can also be realized by recognizing that the given relation corresponds to equality in Cauchy–Schwarz inequality
\[ (6^2 + 3^2 + 2^2)\left[u^2 + (2v)^2 + (3w)^2\right] \geq (6 \cdot u + 3 \cdot 2v + 2 \cdot 3w)^2. \]

After multiplying by \( r \), we see that
\[
\frac{s - a}{36} = \frac{s - b}{9} = \frac{s - c}{4} = \frac{2s - b - c}{9 + 4} = \frac{2s - c - a}{4 + 36} = \frac{2s - a - b}{36 + 9} = \frac{a}{13} = \frac{b}{40} = \frac{c}{45};
\]
that is, triangle \( ABC \) is similar to a triangle with side lengths 13, 40, 45.

**Note:** The technique of using
\[ \frac{a}{b} = \frac{c}{d} = \frac{a + c}{b + d} \]
is rather tricky. However, by Introductory Problem 19(a), we can have
\[ u + v + w = uvw. \]

Since \( u : v : w = 1 : \frac{1}{4} : \frac{1}{9} \), it follows that \( u = 7, v = \frac{7}{4}, \) and \( w = \frac{7}{9} \). Hence by the double-angle formulas, \( \sin A = \frac{13}{25}, \sin B = \frac{56}{65}, \) and \( \sin C = \frac{63}{65}, \) or
\[ \sin A = \frac{13}{25}, \quad \sin B = \frac{40}{325}, \quad \sin C = \frac{45}{325}. \]

By the extended law of sines, triangle \( ABC \) is similar to triangle \( T \) with side lengths 13, 40, and 45. (The circumcircle of \( T \) has diameter \( \frac{325}{7} \).)

18. [USAMO 1996] Prove that the average of the numbers
\[ 2 \sin 2^\circ, \ 4 \sin 4^\circ, \ 6 \sin 6^\circ, \ \ldots, \ 180 \sin 180^\circ, \]
is cot \( 1^\circ \).
First Solution: We need to prove that
\[ 2 \sin 2^\circ + 4 \sin 4^\circ + \cdots + 178 \sin 178^\circ = 90 \cot 1^\circ, \]
which is equivalent to
\[
2 \sin 2^\circ \cdot \sin 1^\circ + 2(2 \sin 4^\circ \cdot \sin 1^\circ) + \cdots + 89(2 \sin 178^\circ \cdot \sin 1^\circ)
= 90 \cos 1^\circ.
\]
Note that
\[ 2 \sin k^\circ \sin 1^\circ = \cos (2k - 1)^\circ - \cos (2k + 1)^\circ. \]
We have
\[
2 \sin 2^\circ \cdot \sin 1^\circ + 2(2 \sin 4^\circ \cdot \sin 1^\circ) + \cdots + 89(2 \sin 178^\circ \cdot \sin 1^\circ)
= (\cos 1^\circ - \cos 3^\circ) + 2(\cos 3^\circ - \cos 5^\circ)
+ \cdots + 89(\cos 177^\circ - \cos 179^\circ)
= \cos 1^\circ + \cos 3^\circ + \cdots + \cos 177^\circ - 89 \cos 179^\circ
= \cos 1^\circ + (\cos 3^\circ + \cos 177^\circ) + \cdots + (\cos 89^\circ + \cos 91^\circ)
+ 89 \cos 1^\circ
= \cos 1^\circ + 89 \cos 1^\circ = 90 \cos 1^\circ,
\]
as desired.

Note: The techniques of telescoping sum and pairing of summands involved in the first solution is rather tricky. The second solution involves complex numbers. It is slightly longer than the first solution. But for the reader who is familiar with the rules of operation for complex numbers and geometric series, every step is natural.

Second Solution: Set the complex number \( z = \cos 2^\circ + i \sin 2^\circ \). Then, by de Moivre’s formula, we have \( z^n = \cos 2n^\circ + i \sin 2n^\circ \). Let \( a \) and \( b \) be real numbers such that
\[ z + 2z^2 + \cdots + 89z^{89} = a + bi. \]
Because \( \sin 180^\circ = 0 \),
\[ b = \frac{1}{2} (2 \sin 2^\circ + 4 \sin 4^\circ + \cdots + 178 \sin 178^\circ + 180 \sin 180^\circ), \]
and it suffices to show that \( b = 45 \cot 1^\circ \).
Set

\[ p_n(x) = x + 2x^2 + \cdots + nx^n. \]

Then

\[ (1 - x)p_n(x) = p_n(x) - xp_n(x) = x + x^2 + \cdots + x^n - nx^{n+1}. \]

Set

\[ q_n(x) = (1 - x)p_n(x) + nx^{n+1} = x + x^2 + \cdots + x^n. \]

Then \( (1 - x)q_n(x) = q_n(x) - xq_n(x) = x - x^{n+1} \). Consequently, we have

\[ p_n(x) = \frac{q_n(x)}{1 - x} - \frac{nx^{n+1}}{1 - x} = \frac{x - x^{n+1}}{(1 - x)^2} - \frac{nx^{n+1}}{1 - x}. \]

It follows that

\[ a + bi = z + 2z^2 + \cdots + 89z^{89} = p_{89}(z) = \frac{z - z^{90}}{(1 - z)^2} - \frac{89z^{90}}{1 - z} = \frac{z + 1}{(z - 1)^2} - \frac{89}{z - 1}, \]

because \( z^{90} = \cos 180^\circ + i \sin 180^\circ = -1 \). Note that \( z + 1 = \text{cis} 2^\circ + \text{cis} 0^\circ = 2 \cos 1^\circ \text{ cis } 1^\circ \) and \( z - 1 = \text{cis} 2^\circ - \text{cis} 0^\circ = 2 \sin 1^\circ \text{ cis } 91^\circ \), and so

\[ a + bi = \frac{2 \cos 1^\circ \text{ cis } 1^\circ}{(2 \sin 1^\circ \text{ cis } 91^\circ)^2} = \frac{89}{2 \sin 1^\circ \text{ cis } 91^\circ} = \frac{2 \cos 1^\circ \text{ cis } 1^\circ}{4 \sin^2 1^\circ \text{ cis } 182^\circ} = \frac{2 \sin 1^\circ}{\cos 1^\circ \text{ cis } (-181^\circ)} = \frac{2 \sin 1^\circ}{2 \sin 1^\circ}. \]

Therefore,

\[ b = \frac{\cos 1^\circ \sin(-181^\circ)}{2 \sin^2 1^\circ} - \frac{89 \sin(-91^\circ)}{2 \sin 1^\circ} = \frac{\cos 1^\circ \sin 1^\circ}{2 \sin^2 1^\circ} + \frac{89 \cos 1^\circ}{2 \sin 1^\circ} = \frac{\cos 1^\circ}{2 \sin 1^\circ} + \frac{89 \cos 1^\circ}{2 \sin 1^\circ} = 45 \cot 1^\circ, \]

as desired.

19. Prove that in any acute triangle \( ABC \),

\[ \cot^3 A + \cot^3 B + \cot^3 C + 6 \cot A \cot B \cot C \geq \cot A + \cot B + \cot C. \]
Solution: Let $\cot A = x$, $\cot B = y$, and $\cot C = z$. Because $xy + yz + zx = 1$ (Introductory Problem 21), it suffices to prove the homogeneous inequality

$$x^3 + y^3 + z^3 + 6xyz \geq (x + y + z)(xy + yz + zx).$$

But this is equivalent to

$$x(x - y)(x - z) + y(y - z)(y - x) + z(z - x)(z - y) \geq 0,$$

which is Schur's inequality.

20. [Turkey 1998] Let $\{a_n\}$ be the sequence of real numbers defined by $a_1 = t$ and $a_{n+1} = 4a_n(1 - a_n)$ for $n \geq 1$. For how many distinct values of $t$ do we have $a_{1998} = 0$?

Solution: Let $f(x) = 4x(1-x) = 1 - (2x - 1)^2$. Observe that if $0 \leq f(x) \leq 1$, then $0 \leq x \leq 1$. Hence if $a_{1998} = 0$, then we must have $0 \leq t \leq 1$. Now choose $0 \leq \theta \leq \frac{\pi}{2}$ such that $\sin \theta = \sqrt{t}$. Observe that for any $\phi \in \mathbb{R}$,

$$f(\sin^2 \phi) = 4 \sin^2 \phi (1 - \sin^2 \phi) = 4 \sin^2 \phi \cos^2 \phi = \sin^2 2\phi;$$

since $a_1 = \sin^2 \theta$, it follows that

$$a_2 = \sin^2 2\theta, \ a_3 = \sin^2 4\theta, \ldots, \ a_{1998} = \sin^2 2^{1997}\theta.$$

Therefore, $a_{1998} = 0$ if and only if $\sin 2^{1997}\theta = 0$. That is, $\theta = \frac{k\pi}{2^{1997}}$ for some integers $k$, and so the values of $t$ for which $a_{1998} = 0$ are $\sin^2(k\pi/2^{1997})$, where $k \in \mathbb{Z}$. Therefore we get $2^{1996} + 1$ such values of $t$, namely, $\sin^2(k\pi/2^{1997})$ for $k = 0, 1, 2, \ldots, 2^{1996}$.

21. Triangle $ABC$ has the following property: there is an interior point $P$ such that $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$, and $\angle PAC = 40^\circ$. Prove that triangle $ABC$ is isosceles.

Solution: Consider Figure 5.5, in which all angles are in degrees.

![Figure 5.5](image-url)
Let $x = \angle PCB$ (in degrees). Then $\angle PBC = 80^\circ - x$. By the law of sines or by Ceva’s theorem,

$$1 = \frac{PA}{PB} \cdot \frac{PB}{PC} \cdot \frac{PC}{PA} = \frac{\sin \angle PBA}{\sin \angle PAB} \cdot \frac{\sin \angle PCB}{\sin \angle PBC} \cdot \frac{\sin \angle PCA}{\sin \angle PCA} = \frac{\sin 20^\circ \sin x \sin 40^\circ}{\sin(80^\circ - x) \sin 30^\circ}.$$ 

The product-to-sum formulas yield

$$1 = \frac{2 \sin x(\sin 30^\circ + \sin 50^\circ)}{\sin(80^\circ - x)} = \frac{\sin x(1 + 2 \cos 40^\circ)}{\sin(80^\circ - x)},$$

and so

$$2 \sin x \cos 40^\circ = \sin(80^\circ - x) - \sin x = 2 \sin(40^\circ - x) \cos 40^\circ,$$

by the difference-to-product formulas. We conclude that $x = 40^\circ - x$, or $x = 20^\circ$. It follows that $\angle ACB = 50^\circ = \angle BAC$, and so triangle $ABC$ is isosceles.

22. Let $a_0 = \sqrt{2} + \sqrt{3} + \sqrt{6}$, and let $a_{n+1} = \frac{a_n^2 - 5}{2(a_n + 2)}$ for integers $n > 0$. Prove that

$$a_n = \cot \left( \frac{2^{n-3} \pi}{3} \right) - 2$$

for all $n$.

**Solution:** By either the double-angle or the half-angle formulas, we obtain

$$\cot \frac{\pi}{24} = \frac{\cos \frac{\pi}{24}}{\sin \frac{\pi}{24}} = \frac{2 \cos^2 \frac{\pi}{24}}{2 \sin \frac{\pi}{24} \cos \frac{\pi}{24}} = \frac{1 + \cos \frac{\pi}{12}}{\sin \left( \frac{\pi}{3} - \frac{\pi}{4} \right)} = \frac{1 + \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4}}{\sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4}}$$

$$= \frac{1 + \sqrt{2} + \sqrt{6}}{\sqrt{6} - \sqrt{2}} = \frac{4 + \sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}}$$

$$= \frac{4(\sqrt{6} + \sqrt{2}) + (\sqrt{6} + \sqrt{2})^2}{(\sqrt{6} - \sqrt{2})(\sqrt{6} + \sqrt{2})} = \frac{4(\sqrt{6} + \sqrt{2}) + 8 + 4\sqrt{3}}{4}$$

$$= 2 + \sqrt{2} + \sqrt{3} + \sqrt{6} = a_0 + 2.$$

Hence $a_n = \cot \left( \frac{2^{n-3} \pi}{3} \right) - 2$ is true for $n = 0$. 

It suffices to show that $b_n = \cot \left( \frac{2^{n-3}\pi}{3} \right)$, where $b_n = a_n + 2$, $n \geq 1$. The recursive relation becomes

$$b_{n+1} - 2 = \frac{(b_n - 2)^2 - 5}{2b_n},$$

or

$$b_{n+1} = \frac{b_n^2 - 1}{2b_n}.$$

Assuming, inductively, that $b_k = \cot c_k$, where $c_k = \frac{2^{k-3}\pi}{3}$, yields

$$b_{k+1} = \frac{\cot^2 c_k - 1}{2 \cot c_k} = \cot 2c_k = \cot c_{k+1},$$

and we are done.

23. [APMC 1982] Let $n$ be an integer with $n \geq 2$. Prove that

$$\prod_{k=1}^{n} \tan \left[ \frac{\pi}{3} \left( 1 + \frac{3^k}{3^n - 1} \right) \right] = \prod_{k=1}^{n} \cot \left[ \frac{\pi}{3} \left( 1 - \frac{3^k}{3^n - 1} \right) \right].$$

**Solution:** Let

$$u_k = \tan \left[ \frac{\pi}{3} \left( 1 + \frac{3^k}{3^n - 1} \right) \right] \quad \text{and} \quad v_k = \tan \left[ \frac{\pi}{3} \left( 1 - \frac{3^k}{3^n - 1} \right) \right].$$

The desired equality becomes

$$\prod_{k=1}^{n} u_k v_k = 1. \quad (\ast)$$

Set

$$t_k = \tan \frac{3^{k-1}\pi}{3^n - 1}.$$ 

Applying the **addition and subtraction formulas** yields

$$u_k = \tan \left( \frac{\pi}{3} + \frac{3^{k-1}\pi}{3^n - 1} \right) = \frac{\sqrt{3} + t_k}{1 - \sqrt{3}t_k} \quad \text{and} \quad v_k = \frac{\sqrt{3} - t_k}{1 + \sqrt{3}t_k}.$$

The **triple-angle formulas** give

$$t_{k+1} = \frac{3t_k - t_k^3}{1 - 3t_k^2},$$
implying that
\[
\frac{t_{k+1}}{t_k} = \frac{3 - t_k^2}{1 - 3t_k^2} = \frac{\sqrt{3} + t_k}{1 - \sqrt{3}t_k} \cdot \frac{\sqrt{3} - t_k}{1 + \sqrt{3}t_k} = u_kv_k.
\]
Consequently,
\[
\prod_{k=1}^n (u_kv_k) = \frac{t_2}{t_1} \cdot \frac{t_3}{t_2} \cdots \frac{t_{n+1}}{t_n} = \frac{\tan\left(\pi + \frac{\pi}{3^{n-1}}\right)}{\tan\left(\frac{\pi}{3^{n-1}}\right)} = 1,
\]
establishing equation (*)

24. [China 1999, by Yuming Huang] Let \(P_2(x) = x^2 - 2\). Find all sequences of polynomials \(\{P_k(x)\}_{k=1}^{\infty}\) such that \(P_k(x)\) is monic (that is, with leading coefficient 1), has degree \(k\), and \(P_i(P_j(x)) = P_j(P_i(x))\) for all positive integers \(i\) and \(j\).

Solution: First, we show that the sequence, if it exists, is unique. In fact, for each \(n\), there can be only one \(P_n\) that satisfies \(P_n(P_2(x)) = P_2(P_n(x))\). Let
\[
P_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.
\]
By assumption,
\[
(x^2 - 2)^n + a_{n-1}(x^2 - 2)^{n-1} + \cdots + a_1(x^2 - 2) + a_0 = (x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)^2 - 2.
\]
Consider the coefficients on both sides. On the left side, \(2i\) is the highest power of \(x\) in which \(a_i\) appears. On the right side, the highest power is \(x^{n+i}\), and there it appears as \(2a_i x^{n+i}\). Thus, we see that the maximal power of \(a_i\) always is higher on the right side. It follows that we can solve for each \(a_i\) in turn, from \(n - 1\) to 0, by equating coefficients. Furthermore, we are guaranteed that the polynomial is unique, since the equation we need to solve to find each \(a_i\) is linear.

Second, we define \(P_n\) explicitly. We claim that \(P_n(x) = 2T_n\left(\frac{x}{2}\right)\) (where \(T_n\) is the \(n\)th Chebyshev polynomial defined in Introductory Problem 49). That is,\(P_n\) is defined by the recursive relation \(P_1(x) = x\), \(P_2(x) = x^2 - 2\), and
\[
P_{n+1}(x) = xP_n(x) - P_{n-1}(x).
\]
Because $T_n(\cos \theta) = \cos n\theta$, $P_n(2 \cos \theta) = 2 \cos n\theta$. It follows that for all $\theta$,

$$P_m(P_n(2 \cos \theta)) = P_m(2 \cos n\theta) = 2 \cos mn\theta$$

$$= P_n(2 \cos m\theta) = P_n(P_m(2 \cos \theta)).$$

Thus, $P_m(P_n(x))$ and $P_n(P_m(x))$ agree at all values of $x$ in the interval $[-2, 2]$. Because both are polynomials, it follows that they are equal for all $x$, which completes the proof.

25. [China 2000, by Xuanguo Huang] In triangle $ABC$, $a \leq b \leq c$. As a function of angle $C$, determine the conditions under which $a + b - 2R - 2r$ is positive, negative, or zero.

Solution: In Figure 5.6, set $\angle A = 2x$, $\angle B = 2y$, $\angle C = 2z$. Then $0 < x \leq y \leq z$ and $x + y + z = \frac{\pi}{2}$. Let $s$ denote the given quantity $a + b - 2R - 2r$. Using the extended law of sines and by Introductory Problem 25(d), we have

$$s = 2R(\sin 2x + \sin 2y - 1 - 4 \sin x \sin y \sin z).$$

Note that in a right triangle $ABC$ with $\angle C = \frac{\pi}{2}$, we have $2R = c$ and $2r = a + b - c$, implying that $s = 0$. Hence, we factor $\cos 2z$ from our expression for $s$. By the sum-to-product, product-to-sum, and double-angle
formulas, we have
\[
\frac{s}{2R} = 2 \sin(x + y) \cos(x - y) - 1 + 2(\cos(x + y) - \cos(x - y) ) \sin z
\]
\[
= 2 \cos z \cos(x - y) - 1 + 2(\sin z - \cos(x - y)) \sin z
\]
\[
= 2 \cos(x - y)(\cos z - \sin z) - \cos 2z
\]
\[
= 2 \cos(y - x) \frac{\cos^2 z - \sin^2 z}{\cos z + \sin z} - \cos 2z
\]
\[
= \left[ \frac{2 \cos(y - x)}{\cos z + \sin z} - 1 \right] \cos 2z,
\]
where we may safely introduce the quantity \( \cos z + \sin z \) because it is positive when \( 0 < z < \frac{\pi}{2} \).

Observe that \( 0 \leq y - x < \min \{ y, x + y \} \leq \min \{ z, \frac{\pi}{2} - z \} \). Because \( z \leq \frac{\pi}{2} \) and \( \frac{\pi}{2} - z \leq \frac{\pi}{2} \), we have \( \cos(y - x) > \max \{ \cos z, \cos \left( \frac{\pi}{2} - z \right) \} = \max \{ \cos z, \sin z \} \). Hence \( 2 \cos(y - x) > \cos z + \sin z \), or
\[
\frac{2 \cos(y - x)}{\cos z + \sin z} - 1 > 0.
\]
Thus, \( s = p \cos 2z \) for some \( p > 0 \). It follows that \( s = a + b - 2R - 2r \) is positive, zero, or negative if and only if angle \( C \) is acute, right, or obtuse, respectively.

26. Let \( ABC \) be a triangle. Points \( D, E, F \) are on sides \( BC, CA, AB \), respectively, such that \( |DC| + |CE| = |EA| + |AF| = |FB| + |BD| \). Prove that
\[
|DE| + |EF| + |FD| \geq \frac{1}{2}(|AB| + |BC| + |CA|).
\]

**Solution:** As shown in Figure 5.7, Let \( E_1 \) and \( F_1 \) be the feet of the perpendicular line segments from \( E \) and \( F \) to line \( BC \).
We have

\[ |EF| \geq |E_1F_1| = a - (|BF| \cos B + |CE| \cos C). \]

Likewise, we have

\[ |DE| \geq c - (|AE| \cos A + |BD| \cos B) \]

and

\[ |FD| \geq b - (|CD| \cos C + |AF| \cos A). \]

Note that \( |DC| + |CE| = |EA| + |AF| = |FB| + |BD| = \frac{1}{3}(a + b + c) \).

Adding the last three inequalities gives

\[ |DE| + |EF| + |FD| \]

\[ \geq a + b + c - \frac{1}{3}(a + b + c)(\cos A + \cos B + \cos C) \]

\[ \geq \frac{1}{2}(a + b + c), \]

by Introductory Problem 27(b). Equality holds if and only if the length of segment \( EF, FD \) and \( DE \) is equal to the length of the projection of segment \( EF \) on line \( BC \) (\( FD \) on \( CA \) and \( DE \) on \( AB \)), and \( A = B = C = 60^\circ \), that is, if and only if \( D, E, \) and \( F \) are the midpoints of an equilateral triangle.

27. Let \( a \) and \( b \) be positive real numbers. Prove that

\[ \frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} \geq \frac{2}{\sqrt{1+ab}} \]

if either (1) \( 0 < a, b \leq 1 \) or (2) \( ab \geq 3 \).


Solution: Because \( a \) and \( b \) are positive real numbers, there are angles \( x \) and \( y \), with \( 0 < x, y < 90^\circ \), such that \( \tan x = a \) and \( \tan y = b \). The desired inequality is clearly true when \( a = b \). Hence we assume that \( a \neq b \), or equivalently, \( x \neq y \). Then \( 1 + a^2 = \sec^2 x \) and \( \frac{1}{\sqrt{1+a^2}} = \cos x \). Note that

\[ 1 + ab = \frac{\cos x \cos y + \sin x \sin y}{\cos x \cos y} = \frac{\cos(x - y)}{\cos x \cos y} \]

by the addition and subtraction formulas. The desired inequality reduces to

\[ \cos x + \cos y \geq 2\sqrt{\frac{\cos x \cos y}{\cos(x - y)}}. \quad (*) \]
To establish part (1), we rewrite inequality \((*)\) as
\[
\cos^2 x + \cos^2 y + 2 \cos x \cos y \leq \frac{4 \cos x \cos y}{\cos(x - y)}.
\]
Because \(0 < |x - y| < 90^\circ\), it follows that \(0 < \cos(x - y) < 1\). Hence
\[
2 \cos x \cos y \leq \frac{2 \cos x \cos y}{\cos(x - y)}.
\]
It suffices to show that
\[
\cos(x - y) \left[ \cos^2 x + \cos^2 y \right] \leq 2 \cos x \cos y,
\]
or
\[
\cos(x - y) \left[ \cos^2 x + \cos^2 y + 2 \right] \leq 4 \cos x \cos y
\]
by the double-angle formulas. By the sum-to-product formulas, the last inequality is equivalent to
\[
\cos(x - y)[2 \cos(x - y) \cos(x + y) + 2] \leq 2[\cos(x - y) + \cos(x + y)],
\]
or \(\cos^2(x - y) \cos(x + y) \leq \cos(x + y)\), which is clearly true, because for \(0 < a, b \leq 1\), we have \(0^\circ < x, y < 45^\circ\), and so \(0^\circ < x + y \leq 90^\circ\) and \(\cos(x + y) > 0\). This completes the proof of part (1).

To prove part (2), we rewrite inequality \((*)\) as
\[
2 \cos \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right) \geq 2 \sqrt{\frac{1}{2} \left[ \cos(x + y) + \cos(x - y) \right]} \cos(x - y)
\]
by the sum-to-product and product-to-sum formulas. Squaring both sides of the inequality and clearing denominators gives
\[
4 \cos^2 \left( \frac{x + y}{2} \right) \cos^2 \left( \frac{x - y}{2} \right) \cos(x - y) \geq 2[\cos(x + y) + \cos(x - y)],
\]
or
\[
[1 + \cos(x + y)][1 + \cos(x - y)] \cos(x - y) \geq 2[\cos(x + y) + \cos(x - y)]
\]
between the double-angle formulas. Setting \(s = \cos(x + y)\) and \(t = \cos(x - y)\), it suffices to prove that
\[
(1 + s)(1 + t)t \geq 2(s + t),
\]
or,
\[
0 \leq (1 + s)t^2 + (s - 1)t - 2s = (t - 1)[(1 + s)t + 2s].
\]
Because \(t \leq 1\), it suffices to show that
\[
(1 + s)t + 2s \leq 0.
\]
Because \( ab \geq 3 \), \( \tan x \tan y \geq 3 \), or equivalently, \( \sin x \sin y \geq 3 \cos x \cos y \).

By the product-to-sum formulas, we have

\[
\frac{1}{2} [\cos(x - y) - \cos(x + y)] \geq \frac{3}{2} [\cos(x - y) + \cos(x + y)],
\]

or \( t \leq -2s \). Because \( 1 + s \geq 0 \), \( (1 + s)t \leq -(1 + s)2s \). Consequently, \( (1 + s)t + 2s \leq -(1 + s)2s + 2s = -2s^2 \leq 0 \), as desired.

28. [China 1998, by Xuanguo Huang] Let \( ABC \) be a nonobtuse triangle such that \( AB > AC \) and \( \angle B = 45^\circ \). Let \( O \) and \( I \) denote the circumcenter and incenter of triangle \( ABC \), respectively. Suppose that \( \sqrt{2} |OI| = |AB| - |AC| \). Determine all the possible values of \( \sin A \).

First Solution: Applying the extended law of sines to triangle \( ABC \) yields \( a = 2R \sin A \), \( b = 2R \sin B \), and \( c = 2R \sin C \). If incircle is tangent to side \( AB \) at \( D \) (Figure 5.8). Then \( |BD| = \frac{c + a - b}{2} \), and so \( r = |ID| = |BD| \tan \frac{B}{2} \). The half-angle formulas give

\[
\tan \frac{B}{2} = \frac{1 - \cos B}{\sin B} = \frac{1 - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \sqrt{2} - 1,
\]

and so

\[
r = R \left( \sqrt{2} - 1 \right) (\sin A + \sin C - \sin B).
\]

By Euler’s formula, \( |OI|^2 = R(R - 2r) \), so we have

\[
|OI|^2 = R^2 - 2Rr = R^2 \left[ 1 - 2(\sin A + \sin C - \sin B) \left( \sqrt{2} - 1 \right) \right].
\]

Squaring both sides of the given equation \( \sqrt{2} |OI| = |AB| - |AC| \) gives

\[
|OI|^2 = \frac{(c - b)^2}{2} = 2R^2(\sin C - \sin B)^2.
\]

Therefore

\[
2(\sin C - \sin B)^2 = 1 - 2(\sin A + \sin C - \sin B) \left( \sqrt{2} - 1 \right)
\]

or

\[
1 - 2 \left( \sin C - \frac{\sqrt{2}}{2} \right)^2 = 2 \left( \sin A + \sin C - \frac{\sqrt{2}}{2} \right) \left( \sqrt{2} - 1 \right). \tag{*}
\]
The addition and subtraction formulas give

\[
\sin C = \sin(180^\circ - B - A) = \sin(135^\circ - A) = \sin 135^\circ \cos A - \cos 135^\circ \sin A = \frac{\sqrt{2}(\sin A + \cos A)}{2},
\]

and so

\[
\sin C - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} (\sin A + \cos A - 1).
\]

Plugging the last equation into equation (*) yields

\[
1 - (\sin A + \cos A - 1)^2 = 2 \left( \sqrt{2} - 1 \right) \left[ \sin A + \frac{\sqrt{2}}{2} (\sin A + \cos A - 1) \right].
\]

Expanding both sides of the last equation gives

\[
1 - (\sin A + \cos A)^2 + 2(\sin A + \cos A) - 1
= \left( \sqrt{2} - 1 \right) \left( 2 + \sqrt{2} \right) \sin A + \left( 2 - \sqrt{2} \right) (\cos A - 1)
\]

or

\[
\sin^2 A + \cos^2 A + 2 \sin A \cos A = \left( 2 - \sqrt{2} \right) \sin A + \sqrt{2} \cos A + \left( 2 - \sqrt{2} \right).
\]

Consequently, we have

\[
2 \sin A \cos A - \left( 2 - \sqrt{2} \right) \sin A - \sqrt{2} \cos A + \left( \sqrt{2} - 1 \right) = 0;
\]

that is,

\[
\left( \sqrt{2} \sin A - 1 \right) \left( \sqrt{2} \cos A - \sqrt{2} + 1 \right) = 0.
\]

This implies that \( \sin A = \frac{\sqrt{2}}{2} \) or \( \cos A = 1 - \frac{\sqrt{3}}{2} \). Therefore, the answer to the problem is

\[
\sin A = \frac{\sqrt{2}}{2} \quad \text{or} \quad \sin A = \sqrt{1 - \cos^2 A} = \frac{\sqrt{4\sqrt{2} - 2}}{2}.
\]
Second Solution: As shown in Figure 5.8, the incircle touches the sides $AB$, $BC$, and $CA$ at $D$, $E$, and $F$, respectively. Let $M$ be the foot of the perpendicular line segments from $O$ to side $BC$. Then the line $OM$ is the perpendicular bisector of $BC$, and $|BM| = |CM|$. From equal tangents, we have $|AF| = |AE|$, $|BD| = |BF|$, and $|CD| = |CE|$. Because $c > b$, $M$ lies on segment $BD$. We find that

$$\sqrt{2}|OI| = c - b = (|AF| + |FB|) - (|AE| + |EC|)$$
$$= |FB| - |EC| = |BD| - |DC|.$$

We deduce that $|BD| = |BM| + |MD|$ and $|DC| = |CM| - |DM|$. Hence $\sqrt{2}|OI| = 2|DM|$, or $|OI| = \sqrt{2}|DM|$. Thus lines $OI$ and $DM$ form a $45^\circ$ angle, which implies that either $OI \perp AB$ or $OI \parallel AB$. We consider these two cases separately.

- **First Case:** In this case, we assume that $OI \perp AB$. Then $OI$ is the perpendicular bisector of side $AB$; that is, the incenter lies on the perpendicular bisector of side $AB$. Thus triangle $ABC$ must be isosceles, with $|AC| = |BC|$, and so $A = B = 45^\circ$ and $\sin A = \frac{\sqrt{2}}{2}$.

- **Second Case:** In this case, we assume that $OI \parallel AB$. Let $N$ be the midpoint of side $AB$. Then $OIFN$ is a rectangle. Note that $\angle AON = \angle C$, and thus

$$R \cos \angle AON = R \cos C = |ON| = |IF| = r.$$

By the solution to Introductory Problem 27, we have

$$\cos C = \frac{R}{r} = \cos A + \cos B + \cos C - 1,$$
implying that \( \cos A = 1 - \cos B = 1 - \sqrt{2}/2 \). It follows that
\[
\sin A = \sqrt{1 - \cos^2 A} = \frac{\sqrt{4\sqrt{2} - 2}}{2}.
\]

29. [Dorin Andrica] Let \( n \) be a positive integer. Find the real numbers \( a_0 \) and \( a_{k,\ell} \), \( 1 \leq \ell < k \leq n \), such that
\[
\frac{\sin^2 nx}{\sin^2 x} = a_0 + \sum_{1 \leq \ell < k \leq n} a_{\ell,k} \cos(2(k - \ell)x)
\]
for all real numbers \( x \) with \( x \) not an integer multiple of \( \pi \).

**Solution:** In this solution, we apply a similar technique to that shown in the first solution of Advanced Problem 18. Note that
\[
2 \sin 2kx \sin x = \cos(2k - 1)x - \cos(2k + 1)x.
\]
We have
\[
2 \sin x (\sin 2x + \sin 4x + \cdots + \sin 2nx) = [\cos x - \cos 3x] + [\cos 3x - \cos 5x] + \cdots + [\cos(2n - 1)x - \cos(2n + 1)x] = \cos x - \cos(2n + 1)x = 2 \sin nx \sin(n + 1)x,
\]
or
\[
s = \sin 2x + \sin 4x + \cdots + \sin 2nx = \frac{\sin nx \sin(n + 1)x}{\sin x}.
\]
Similarly, by noting that
\[
2 \cos 2kx \sin x = \sin(2k + 1)x - \sin(2k - 1)x,
\]
we have
\[
2 \sin x (\cos 2x + \cos 4x + \cdots + \cos 2nx) = [\sin 3x - \sin x] + [\sin 5x - \sin 3x] + \cdots + [\sin(2n + 1)x - \sin(2n - 1)x] = \sin(2n + 1)x - \sin x = 2 \sin nx \cos(n + 1)x,
\]
or
\[
c = \cos 2x + \cos 4x + \cdots + \cos 2nx = \frac{\sin nx \cos(n + 1)x}{\sin x}.
\]
It follows that
\[
\left( \frac{\sin^2 nx}{\sin^2 x} \right)^2 = \left( \frac{\sin nx \sin(n+1)x}{\sin x} \right)^2 + \left( \frac{\sin nx \cos(n+1)x}{\sin x} \right)^2 = s^2 + c^2.
\]

On the other hand,
\[
\begin{align*}
s^2 + c^2 &= (\sin 2x + \sin 4x + \cdots + \sin 2nx)^2 \\
&\quad + (\cos 2x + \cos 4x + \cdots + \cos 2nx)^2 \\
&= n + \sum_{1 \leq \ell < k \leq n} (2 \sin 2\ell x \sin 2kx + 2 \cos 2\ell x \cos 2kx) \\
&= n + 2 \sum_{1 \leq \ell < k \leq n} \cos 2(k-\ell)x
\end{align*}
\]

by the product-to-sum formulas. Setting \(a_0 = n\) and \(a_{\ell, k} = 2\) solves the problem.

30. [USAMO 2000] Let \(S\) be the set of all triangles \(ABC\) for which
\[
5 \left( \frac{1}{|AP|} + \frac{1}{|BQ|} + \frac{1}{|CR|} \right) - \frac{3}{\min \{ |AP|, |BQ|, |CR| \}} = \frac{6}{r},
\]
where \(r\) is the inradius and \(P, Q, \) and \(R\) are the points of tangency of the incircle with sides \(AB, BC, \) and \(CA,\) respectively. Prove that all triangles in \(S\) are isosceles and similar to one another.

**Solution:** Let \(I\) be the incenter of triangle \(ABC.\) Then \(|IP| = |IQ| = |IR| = r.\) By symmetry, we may assume that \(\min \{ |AP|, |BQ|, |CR| \} = |AP|,\) as shown in Figure 5.9. Let \(x = \tan \frac{A}{2}, y = \tan \frac{B}{2}, \) and \(z = \tan \frac{C}{2}.\) By Introductory Problem 19(a), we also have
\[
xy + yz + zx = 1. \quad (*)
\]
Note that $|AP| = \frac{x}{y}$, $|BQ| = \frac{y}{z}$, and $|CR| = \frac{z}{x}$. Then the equation given in the problem statement becomes

$$2x + 5y + 5z = 6. \quad (***)$$

Eliminating $x$ from equations $(\ast)$ and $(**)$ yields

$$5y^2 + 5z^2 - 8yz - 6y - 6z + 2 = 0.$$ 

Completing the squares, we obtain

$$(3y - 1)^2 + (3z - 1)^2 = 4(y - z)^2.$$ 

Setting $3y - 1 = u$ and $3z - 1 = v$ gives $y = \frac{u+1}{3}$ and $z = \frac{v+1}{3}$, and so $y - z = \frac{u-v}{3}$. The above equation becomes

$$5u^2 + 8uv + 5v^2 = 0.$$ 

Because the discriminant of this quadratic equation is $8^2 - 4 \cdot 5 \cdot 25 < 0$, the only real solution to the equation is $u = v = 0$. Thus there is only one possible set of values for the tangents of half-angles of $ABC$ (namely, $x = \frac{4}{3}$ and $y = z = \frac{1}{3}$). Thus all triangles in $S$ are isosceles and similar to one another.

Indeed, we have $x = \frac{r}{|AP|} = \frac{4}{3}$ and $y = z = \frac{r}{|BQ|} = \frac{r}{|CR|} = \frac{1}{3} = \frac{4}{12}$, so we can set $r = 4$, $|AP| = |AR| = 3$, and $|BP| = |BQ| = |CQ| = |CR| = 12$. This leads to $|AB| = |AC| = 15$ and $|BC| = 24$. By scaling, all triangles in $S$ are similar to the triangle with side lengths 5, 5, 8.

We can also use the half-angle formulas to calculate

$$\sin B = \sin C = \frac{2 \tan \frac{C}{2}}{1 + \tan^2 \frac{C}{2}} = \frac{3}{5}.$$ 

From this it follows that $|AQ| : |QB| : |BA| = 3 : 4 : 5$ and $|AB| : |AC| : |BC| = 5 : 5 : 8$.

31. [TST 2003] Let $a, b, c$ be real numbers in the interval $(0, \frac{\pi}{2})$. Prove that

$$\frac{\sin a \sin(a - b) \sin(a - c)}{\sin(b + c)} + \frac{\sin b \sin(b - c) \sin(b - a)}{\sin(c + a)} + \frac{\sin c \sin(c - a) \sin(c - b)}{\sin(a + b)} \geq 0.$$
Solution: By the product-to-sum and the double-angle formulas, we have
\[
\sin(\alpha - \beta) \sin(\alpha + \beta) = \frac{1}{2} \left[ \cos 2\beta - \cos 2\alpha \right] \\
= \sin^2 \alpha - \sin^2 \beta.
\]
Hence, we obtain
\[
\sin a \sin(a - b) \sin(a - c) \sin(a + b) \sin(a + c) = \sin a \left( \sin^2 a - \sin^2 b \right) \left( \sin^2 a - \sin^2 c \right)
\]
and its analogous cyclic symmetric forms. Therefore, it suffices to prove that
\[
x \left( x^2 - y^2 \right) \left( x^2 - z^2 \right) + y \left( y^2 - z^2 \right) \left( y^2 - x^2 \right) + z \left( z^2 - x^2 \right) \left( z^2 - y^2 \right) \geq 0,
\]
where \( x = \sin a, \ y = \sin b, \) and \( z = \sin c \) (hence \( x, y, z > 0 \)). Since the last inequality is symmetric with respect to \( x, y, \) and \( z \), we may assume that \( 0 < x \leq y \leq z \). It suffices to prove that
\[
x \left( y^2 - x^2 \right) \left( z^2 - x^2 \right) + z \left( z^2 - x^2 \right) \left( z^2 - y^2 \right) \geq y \left( z^2 - y^2 \right) \left( y^2 - x^2 \right),
\]
which is evident, because
\[
x \left( y^2 - x^2 \right) \left( z^2 - x^2 \right) \geq 0
\]
and
\[
z \left( z^2 - x^2 \right) \left( z^2 - y^2 \right) \geq z \left( y^2 - x^2 \right) \left( z^2 - y^2 \right) \geq y \left( z^2 - y^2 \right) \left( y^2 - x^2 \right).
\]
Note: The key step of the proof is an instance of Schur’s inequality with \( r = \frac{1}{2} \).

32. [TST 2002] Let \( ABC \) be a triangle. Prove that
\[
\sin \frac{3A}{2} + \sin \frac{3B}{2} + \sin \frac{3C}{2} \leq \cos \frac{A - B}{2} + \cos \frac{B - C}{2} + \cos \frac{C - A}{2}.
\]
First Solution: Let \( \alpha = \frac{A}{2}, \ \beta = \frac{B}{2}, \ \gamma = \frac{C}{2} \). Then \( 0^\circ < \alpha, \beta, \gamma < 90^\circ \) and
\[ \alpha + \beta + \gamma = 90^\circ. \] By the **difference-to-product formulas**, we have

\[
\sin \frac{3A}{2} - \cos \frac{B - C}{2} = \sin 3\alpha - \cos(\beta - \gamma) = \sin 3\alpha - \sin(\alpha + 2\gamma) = 2 \cos(2\alpha + \gamma) \sin(\alpha - \gamma) = -2 \sin(\alpha - \beta) \sin(\alpha - \gamma).
\]

In exactly the same way, we can show that

\[
\sin \frac{3B}{2} - \cos \frac{C - A}{2} = -2 \sin(\beta - \alpha) \sin(\beta - \gamma)
\]

and

\[
\sin \frac{3C}{2} - \cos \frac{A - B}{2} = -2 \sin(\gamma - \alpha) \sin(\gamma - \beta).
\]

Hence it suffices to prove that

\[
\sin(\alpha - \beta) \sin(\alpha - \gamma) + \sin(\beta - \alpha) \sin(\beta - \gamma) + \sin(\gamma - \alpha) \sin(\gamma - \beta) \geq 0.
\]

Note that this inequality is symmetric with respect to \(\alpha, \beta,\) and \(\gamma,\) so we can assume without loss of generality that \(0^\circ < \alpha \leq \beta \leq \gamma < 90^\circ.\) Then regrouping the terms on the left-hand side gives

\[
\sin(\alpha - \beta) \sin(\alpha - \gamma) + \sin(\gamma - \alpha) \sin(\gamma - \beta) = -\sin(\beta - \alpha) \sin(\beta - \gamma) - \sin(\alpha - \beta) \sin(\alpha - \gamma) + \sin(\gamma - \alpha) \sin(\gamma - \beta).
\]

which is positive because the function \(y = \sin x\) is increasing for \(0^\circ < x < 90^\circ.\)

**Note:** Again the proof is similar to that of **Schur’s inequality**.

**Second Solution:** We maintain the same notation as in the first solution. By the **addition and subtraction formulas**, we have

\[
\sin 3\alpha = \sin \alpha \cos 2\alpha + \sin 2\alpha \cos \alpha;
\]
\[
\cos(\beta - \alpha) = \sin(2\alpha + \gamma) = \sin 2\alpha \cos \gamma + \sin \gamma \cos 2\alpha;
\]
\[
\cos(\beta - \gamma) = \sin(2\gamma + \alpha) = \sin 2\gamma \cos \alpha + \sin \alpha \cos 2\gamma;
\]
\[
\sin 3\gamma = \sin \gamma \cos 2\gamma + \sin 2\gamma \cos \gamma.
\]
By the difference-to-product formulas, it follows that
\[ \sin 3\alpha + \sin 3\gamma - \cos(\beta - \alpha) - \cos(\beta - \gamma) = (\sin \alpha - \sin \gamma)(\cos 2\alpha - \cos 2\gamma) + (\cos \alpha - \cos \gamma)(\sin 2\alpha - \sin 2\gamma) + 2(\cos \alpha - \cos \gamma) \cos(\alpha + \gamma) \sin(\alpha - \gamma). \]

Note that \(\sin x\) is increasing, and \(\cos x\) and \(\cos 2x\) are decreasing for \(0^\circ < x < 90^\circ\). Since \(0^\circ < \alpha, \gamma, \alpha + \gamma < 90^\circ\), each of the two products in the last sum is less than or equal to 0. Hence
\[ \sin 3\alpha + \sin 3\gamma - \cos(\beta - \alpha) - \cos(\beta - \gamma) \leq 0. \]

In exactly the same way, we can show that
\[ \sin 3\beta + \sin 3\alpha - \cos(\gamma - \beta) - \cos(\gamma - \alpha) \leq 0 \]
and
\[ \sin 3\gamma + \sin 3\beta - \cos(\alpha - \gamma) - \cos(\alpha - \beta) \leq 0. \]

Adding the last three inequalities gives the desired result.

33. Let \(x_1, x_2, \ldots, x_n, n \geq 2\), be \(n\) distinct real numbers in the interval \([-1, 1]\).
Prove that
\[ \frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} \geq 2^{n-2}, \]
where \(t_i = \prod_{j \neq i} |x_j - x_i|\).

**Solution:** Let \(T_n\) denote the \(n\)th **Chebyshev polynomial**. Recall that (Introductory Problem 49) \(T_n(\cos x) = \cos nx\) and \(T_n\) is defined by the recursion \(T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)\). \(T_0(x) = 1\), and \(T_1(x) = x\). Therefore, the leading coefficient of \(T_n\) is \(2^n-1\) for \(n \geq 1\).

Now we apply the above information to the problem at hand. We can apply **Lagrange’s interpolation formula** to the points \(x_1, x_2, \ldots, x_n\) and the polynomial \(T_{n-1}(x)\) to obtain
\[ T_{n-1}(x) = \sum_{k=1}^{n} \frac{T_{n-1}(x_k)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}. \]
Equating leading coefficients, we have

$$2^{n-2} = \sum_{k=1}^{n} \frac{T_{n-1}(x_k)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}.$$ 

Set $\theta_k$ such that $\cos \theta_k = x_k$. Then $|T_{n-1}(x_k)| = |\cos(n-1)\theta_k| \leq 1$. It follows that

$$2^{n-2} \leq \sum_{k=1}^{n} \frac{|T_{n-1}(x_k)|}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)} = \sum_{k=1}^{n} \frac{1}{r_k},$$

as desired.

34. [St. Petersburg 2001] Let $x_1, \ldots, x_{10}$ be real numbers in the interval $[0, \pi/2]$ such that $\sin^2 x_1 + \sin^2 x_2 + \cdots + \sin^2 x_{10} = 1$. Prove that

$$3(\sin x_1 + \cdots + \sin x_{10}) \leq \cos x_1 + \cdots + \cos x_{10}.$$ 

**Solution:** Because $\sin^2 x_1 + \sin^2 x_2 + \cdots + \sin^2 x_{10} = 1$,

$$\cos x_i = \sqrt{\sum_{j \neq i} \sin^2 x_j}.$$ 

By the **power mean inequality**, for each $1 \leq i \leq 10$,

$$\cos x_i = \sqrt{\sum_{j \neq i} \sin^2 x_j} \geq \frac{\sum_{j \neq i} \sin x_j}{\frac{3}{3}}.$$ 

Summing over all the terms $\cos x_i$ gives

$$\sum_{i=1}^{10} \cos x_i \geq \sum_{i=1}^{10} \sum_{j \neq i} \sin x_j = \sum_{i=1}^{10} \frac{9}{3} \cdot \sin x_i = 3 \sum_{i=1}^{10} \sin x_i,$$

as desired.
35. [IMO 2001 shortlist] Let \( x_1, x_2, \ldots, x_n \) be arbitrary real numbers. Prove the inequality
\[
\frac{x_1}{1 + x_1^2} + \frac{x_2}{1 + x_1^2 + x_2^2} + \cdots + \frac{x_n}{1 + x_1^2 + \cdots + x_n^2} < \sqrt{n}.
\]

**Solution:** (By Ricky Liu) We make the following substitutions: \( x_1 = \tan \alpha_1, \) \( x_2 = \sec \alpha_1 \tan \alpha_2, \) and

\[
x_k = \sec \alpha_1 \sec \alpha_2 \cdots \sec \alpha_{k-1} \tan \alpha_k,
\]

with \(-\pi/2 < \alpha_k < \pi/2, 1 \leq k \leq n\). Note that this is always possible because the range of \( \tan \alpha \) is \((-\infty, \infty)\) and \( \sec \alpha \) is always nonzero. Then the \( k \)th term on the left-hand side of our inequality becomes
\[
\sec \alpha_1 \cdots \sec \alpha_{k-1} \tan \alpha_k \frac{1 + \tan^2 \alpha_1 + \cdots + \sec^2 \alpha_1 \cdot \sec^2 \alpha_{n-1} \tan^2 \alpha_n}{1 + \tan^2 \alpha_1 + \cdots + \sec^2 \alpha_1 \cdot \sec^2 \alpha_{n-1} \tan^2 \alpha_n} = \cos \alpha_1 \cos \alpha_2 \cdots \cos \alpha_k \sin \alpha_k.
\]

Hence the given inequality reduces to
\[
\cos \alpha_1 \sin \alpha_1 + \cos \alpha_1 \cos \alpha_2 \sin \alpha_2 + \cdots + \cos \alpha_1 \cos \alpha_2 \cdots \cos \alpha_n \sin \alpha_n < \sqrt{n};
\]

that is,
\[
c_1 s_1 + c_1 c_2 s_2 + \cdots + c_1 c_2 \cdots c_n s_n < \sqrt{n},
\]
where \( c_i = \cos \alpha_i \) and \( s_i = \sin \alpha_i \) for \( 1 \leq i \leq n \). For \( 2 \leq i \leq n \), because \( c_i^2 + s_i^2 = \cos^2 \alpha_i + \sin^2 \alpha_i = 1 \), we note that
\[
c_i^2 c_2^2 \cdots c_{i-1}^2 s_i^2 + c_i^2 c_2^2 \cdots c_{i-1}^2 c_i^2 = c_i^2 c_2^2 \cdots c_{i-1}^2.
\]

Therefore,
\[
s_1^2 + c_1^2 s_2^2 + \cdots + c_1^2 c_2^2 \cdots c_{n-2}^2 s_{n-1}^2 + c_1^2 c_2^2 \cdots c_{n-1}^2 = 1. \quad (*)
\]

By (*) and Cauchy–Schwarz inequality, we obtain
\[
c_1 s_1 + c_1 c_2 s_2 + \cdots + c_1 c_2 \cdots c_n s_n
\]
\[
\leq \sqrt{s_1^2 + c_1^2 s_2^2 + \cdots + c_1^2 c_2^2 \cdots c_{n-2}^2 s_{n-1}^2 + c_1^2 c_2^2 \cdots c_{n-1}^2}
\]
\[
\cdot \sqrt{c_1^2 + c_2^2 + \cdots + c_{n-1}^2 + c_n^2}
\]
\[
= \sqrt{c_1^2 + c_2^2 + \cdots + c_{n-1}^2 + c_n^2}
\]
\[
= \sqrt{\cos^2 \alpha_1 + \cos^2 \alpha_2 + \cdots + \cos^2 \alpha_{n-1} + \cos^2 \alpha_n \sin^2 \alpha_n}
\]
\[
\leq \sqrt{n}.
\]
Equality in the last step can hold only when
\[
\cos \alpha_1 = \cos \alpha_2 = \cdots = \cos \alpha_{n-1} = \cos \alpha_n \sin \alpha_n = 1,
\]
which is impossible, because \( \cos \alpha_n \sin \alpha_n = \frac{1}{2} \sin 2\alpha_n < 1 \). Therefore, we always have strict inequality, and we are done.

36. [USAMO 1998] Let \( a_0, a_1, \ldots, a_n \) be numbers in the interval \( (0, \frac{\pi}{2}) \) such that
\[
\tan \left( a_0 - \frac{\pi}{4} \right) + \tan \left( a_1 - \frac{\pi}{4} \right) + \cdots + \tan \left( a_n - \frac{\pi}{4} \right) \geq n - 1.
\]
Prove that
\[
\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.
\]

Solution: Let \( b_k = \tan \left( a_k - \frac{\pi}{4} \right), \) \( k = 0, 1, \ldots, n. \) It follows from the hypothesis that for each \( k, -1 < b_k < 1, \) and
\[
1 + b_k \geq \sum_{0 \leq \ell \neq k \leq n} (1 - b_\ell). \quad (*)
\]
Applying the arithmetic-geometric means inequality to the positive real numbers \( 1 - b_\ell, \ell = 0, 1, \ldots, k-1, k+1, \ldots, n, \) we obtain
\[
\sum_{0 \leq \ell \neq k \leq n} (1 - b_\ell) \geq n \left( \prod_{0 \leq \ell \neq k \leq n} (1 - b_\ell) \right)^{1/n}. \quad (**)
\]
From inequalities (*) and (**) it follows that
\[
\prod_{k=0}^{n} (1 + b_k) \geq n^{n+1} \left( \prod_{\ell=0}^{n} (1 - b_\ell) \right)^{1/n},
\]
and hence that
\[
\prod_{k=0}^{n} \frac{1 + b_k}{1 - b_k} \geq n^{n+1}.
\]
Because
\[
\frac{1 + b_k}{1 - b_k} = \frac{1 + \tan \left( a_k - \frac{\pi}{4} \right)}{1 - \tan \left( a_k - \frac{\pi}{4} \right)} = \tan \left[ \left( a_k - \frac{\pi}{4} \right) + \frac{\pi}{4} \right] = \tan a_k,
\]
the conclusion follows.
Note: Using a similar method, one can show that

\[ \frac{1}{n - 1 + a_1} + \frac{1}{n - 1 + a_2} + \cdots + \frac{1}{n - 1 + a_n} \leq 1, \]

where \( a_1, a_2, \ldots, a_n \) are positive real numbers such that \( a_1 a_2 \cdots a_n = 1 \).

An interesting exercise is to provide a trigonometry interpretation for the last inequality.

37. [MOSP 2001] Find all triples of real numbers \((a, b, c)\) such that \(a^2 - 2b^2 = 1\), \(2b^2 - 3c^2 = 1\), and \(ab + bc + ca = 1\).

Solution: Since \(a^2 - 2b^2 = 1\), \(a \neq 0\). Since \(2b^2 - 3c^2 = 1\), \(b \neq 0\). If \(c = 0\), then \(b = 1/\sqrt{2}\) and \(a = \sqrt{2}\). It is easy to check that \((a, b, c) = (\sqrt{2}, 1/\sqrt{2}, 0)\) is a solution of the system. We claim that there are no other valid triples.

We approach the problem indirectly by assuming that there exists a triple of real numbers \((a, b, c)\), with \(abc \neq 0\), such that the equations hold. Without loss of generality, we assume that two of the numbers are positive; otherwise, we can consider the triple \((-a, -b, -c)\). Without loss of generality, we assume that \(a\) and \(b\) are positive. (The first two equations are independent of the signs of \(a, b, c\), and the last equation is symmetric with respect to \(a, b, c\).) By Introductory Problem 21, we may assume that \(a = \cot A\), \(b = \cot B\), and \(c = \cot C\), with \(0 < A, B < 90^\circ\), where \(A, B, C\) are angles of a triangle. We have

\[ a^2 + 1 = 2 \left(b^2 + 1\right) = 3 \left(c^2 + 1\right). \]

The last equation reduces to

\[ \csc^2 A = 2 \csc^2 B = 3 \csc^2 C, \]

or

\[ \frac{1}{\sin A} = \frac{\sqrt{2}}{\sin B} = \frac{\sqrt{3}}{\sin C}. \]

By the law of sines, we conclude that the sides opposite angles \(A, B, C\) have lengths \(k, \sqrt{2}k, \sqrt{3}k\), respectively, for some positive real number \(k\). But then triangle \(ABC\) is a right triangle with \(\angle C = 90^\circ\), implying that \(c = \cot C = 0\), a contradiction to the assumption that \(c \neq 0\). Hence our assumption was wrong, and \((a, b, c) = (\sqrt{2}, 1/\sqrt{2}, 0)\) is the only valid triple sought.
38. Let \( n \) be a positive integer, and let \( \theta_i \) be angles with \( 0 < \theta_i < 90^\circ \) such that
\[
\cos^2 \theta_1 + \cos^2 \theta_2 + \cdots + \cos^2 \theta_n = 1.
\]
Prove that
\[
\tan \theta_1 + \tan \theta_2 + \cdots + \tan \theta_n \geq (n - 1)(\cot \theta_1 + \cot \theta_2 + \cdots + \cot \theta_n).
\]

**Solution:** (By Tiankai Liu) By the **power mean inequality**, for positive numbers \( x_1, x_2, \ldots, x_n \), we have \( M_{-1} \leq M_1 \leq M_2 \); that is,
\[
\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}.
\]
For \( 1 \leq i \leq n \), let \( \cos \theta_i = a_i \). Then
\[
\tan \theta_i = \frac{\sin \theta_i}{\cos \theta_i} = \frac{\sqrt{1 - \cos^2 \theta_i}}{a_i} = \frac{\sqrt{a_i^2 + a_1^2 + \cdots + a_{i-1}^2 + a_{i+1}^2 + \cdots + a_n^2}}{a_i} \\
\geq \frac{a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n}{a_i \sqrt{n - 1}}.
\]
Summing the above inequalities for \( i \) from 1 to \( n \) gives
\[
\sum_{i=1}^{n} \tan \theta_i \geq \frac{1}{\sqrt{n - 1}} \sum_{i=1}^{n} \sum_{j \neq i} \frac{a_j}{a_i} = \frac{1}{\sqrt{n - 1}} \sum_{1 \leq i, j \leq n \atop i \neq j} \frac{a_j}{a_i} , \quad (\ast)
\]
because each ratio \( \frac{a_i}{a_j} \) appears exactly once.

On the other hand, we have
\[
\cot \theta_i = \frac{\cos \theta_i}{\sin \theta_i} = \frac{a_i}{\sqrt{1 - \cos^2 \theta_i}} = \frac{\sqrt{a_i^2 + a_1^2 + \cdots + a_{i-1}^2 + a_{i+1}^2 + \cdots + a_n^2}}{a_i} \\
\leq \frac{a_i \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{i-1}} + \frac{1}{a_{i+1}} + \cdots + \frac{1}{a_n} \right)}{(n - 1) \sqrt{n - 1}},
\]
by the power mean inequality. Summing the above identities from 1 to \(n\) yields

\[
\sum_{i=1}^{n} \cot \theta_i \leq \frac{1}{(n-1)^{3/2}} \sum_{i=1}^{n} \sum_{j \neq i} a_i a_j = \frac{1}{(n-1)^{3/2}} \sum_{1 \leq i, j \leq n, i \neq j} a_i a_j \tag{**}
\]

again, because each ratio \(\frac{a_i}{a_j}\) appears once. Combining inequalities (*) and (**) gives

\[
\sqrt{n-1} \sum_{i=1}^{n} \tan \theta_i \geq \sum_{1 \leq i, j \leq n, i \neq j} \frac{a_j}{a_i} = \sum_{1 \leq i, j \leq n, i \neq j} \frac{a_i}{a_j} \geq (n-1)^{3/2} \sum_{i=1}^{n} \cot \theta_i,
\]

from which the desired result follows.

39. [Weichao Wu] One of the two inequalities

\[
\sin^2 x \cos x < \cos^2 x \cos x \quad \text{and} \quad \sin^2 x > \cos^2 x \cos x
\]

is always true for all real numbers \(x\) such that \(0 < x < \frac{\pi}{4}\). Identify that inequality and prove your result.

**Solution:** The first inequality is true. Observe that the logarithm function is concave down. We apply Jensen’s inequality to the points \(\sin x < \cos x < \sin x + \cos x\) with weights \(\lambda_1 = \tan x\) and \(\lambda_2 = 1 - \tan x\) (because \(0 < x < \pi/4\), \(\lambda_1\) and \(\lambda_2\) are positive) to obtain

\[
\log(\cos x) = \log([\tan x \sin x + (1 - \tan x)(\sin x + \cos x)]) > \tan x \log(\sin x) + (1 - \tan x) \log(\sin x + \cos x).
\]

Since \(\sin x + \cos x = \sqrt{2} \sin \left( x + \frac{\pi}{4} \right) > 1\) and \(\tan x < 1\) in the specified interval, the second term is positive and we may drop it to obtain

\[
\log(\cos x) > \tan x \log(\sin x).
\]

Multiplying by \(\cos x\) and exponentiating gives the required inequality.

40. Let \(k\) be a positive integer. Prove that \(\sqrt{k+1} - \sqrt{k}\) is not the real part of the complex number \(z\) with \(z^n = 1\) for some positive integer \(n\).

**Note:** In June 2003, this problem was first given in the training of the Chinese IMO team, and then in the MOSP. The following solution was due to Anders Kaseorg, gold medalist in the 44th IMO in July 2003 in Tokyo, Japan.
Solution: Assume to the contrary that \( \alpha = \sqrt{k+1} - \sqrt{k} \) is the real part of some complex number \( z \) with \( z^n = 1 \) for some positive integer \( n \). Because \( z \) is an \( n \)th root of unity, it can be written as \( \cos \frac{2\pi j}{n} + i \sin \frac{2\pi j}{n} \) for some integer \( j \) with \( 0 \leq j \leq n-1 \). Thus, \( \alpha = \cos \frac{2\pi j}{n} \).

Let \( T_n(x) \) be the \( n \)th Chebyshev polynomial; that is, \( T_0(x) = 1 \), \( T_1(x) = x \), and \( T_{i+1} = 2xT_i(x) - T_{i-1}(x) \) for \( i \geq 1 \). Then \( T_n(\cos \theta) = \cos (n\theta) \), implying that \( T_n(\alpha) = \cos (2\pi j) = 1 \).

Let \( \beta = \sqrt{k+1} + \sqrt{k} \). Note that \( \alpha \beta = 1 \) and \( \alpha + \beta = 2\sqrt{k+1} \), and so \( \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha \beta = 4k + 2 \). Thus \( \pm \alpha \) and \( \pm \beta \) are the roots of the polynomial

\[
P(x) = (x - \alpha)(x + \alpha)(x - \beta)(x + \beta) = \left(x^2 - \alpha^2\right)\left(x - \beta^2\right) = x^4 - (4k + 2)x + 1.
\]

Let \( Q(x) \) be the minimal polynomial for \( \alpha \). If neither \( \beta \) nor \( -\beta \) is a root of \( Q(x) \), then \( Q(x) \) must divide \( (x - \alpha)(x + \alpha) = x^2 - [2k + 1 - 2\sqrt{k(k+1)}] \), and so \( k(k+1) \) must be a perfect square, which is impossible because \( k^2 < k(k+1) < (k+1)^2 \). Therefore, either \( Q(\beta) \) or \( Q(-\beta) = 0 \) or both. We say that \( Q(\beta') = 0 \), with either \( \beta' = \beta \) or \( \beta' = -\beta \).

Because \( \alpha \) is a root of \( T_n(x) - 1 \), \( Q(x) \) divides \( T_n(x) - 1 \) and \( \beta' \) is a root of \( T_n(x) - 1 \). However, by Introductory Problem 49(f), the roots of \( T_n(x) - 1 \) are all in the interval \([-1, 1]\) and \( |\beta'| = \sqrt{k+1} + \sqrt{k} > 1 \), which is a contradiction. Therefore, our original assumption was wrong, and \( \sqrt{k+1} - \sqrt{k} \) is not the real part of any \( n \)th root of unity.

41. Let \( A_1A_2A_3 \) be an acute-angled triangle. Points \( B_1, B_2, B_3 \) are on sides \( A_2A_3, A_3A_1, A_1A_2 \), respectively. Prove that

\[
2(b_1 \cos A_1 + b_2 \cos A_2 + b_3 \cos A_3) \geq a_1 \cos A_1 + a_2 \cos A_2 + a_3 \cos A_3,
\]

where \( a_i = |A_{i+1}A_{i+2}| \) and \( b_i = |B_{i+1}B_{i+2}| \) for \( i = 1, 2, 3 \) (with indices taken modulo 3; that is, \( x_{i+3} = x_i \)).

Solution: As shown in Figure 5.10, let \( |B_iA_{i+1}| = s_i \) and \( |B_iA_{i+2}| = t_i \), \( i = 1, 2, 3 \). Then \( a_i = s_i + t_i \). Our approach here is similar to that of Advanced Problem 26. Let \( A_1 = \angle A_1, A_2 = \angle A_2, \) and \( A_3 = \angle A_3 \).
Note that segment $EF$, the projection of segment $B_2B_3$ onto the line $A_2A_3$, has length $a_1 - t_3 \cos A_2 - s_2 \cos A_3$, and so

$$b_1 \geq a_1 - t_3 \cos A_2 - s_2 \cos A_3.$$ 

Because $0 < A_1 < 90^\circ$, we know that

$$b_1 \cos A_1 \geq a_1 \cos A_1 - t_3 \cos A_2 \cos A_1 - s_2 \cos A_3 \cos A_1.$$ 

Likewise, we find that

$$b_2 \cos A_2 \geq a_2 \cos A_2 - t_1 \cos A_3 \cos A_2 - s_3 \cos A_1 \cos A_2$$ 

and

$$b_3 \cos A_3 \geq a_3 \cos A_3 - t_2 \cos A_1 \cos A_3 - s_1 \cos A_2 \cos A_3.$$ 

Adding the last three inequalities, we observe that

$$\sum_{i=1}^{3} b_i \cos A_i \geq \sum_{i=1}^{3} a_i (\cos A_i - \cos A_{i+1}A_{i+2}).$$ 

It suffices to show that

$$2 \sum_{i=1}^{3} a_i (\cos A_i - \cos A_{i+1}A_{i+2}) \geq \sum_{i=1}^{3} a_i \cos A_i,$$ 

or

$$\sum_{i=1}^{3} a_i (\cos A_i - 2 \cos A_{i+1}A_{i+2}) \geq 0.$$
Applying the **law of sines** to triangle $A_1 A_2 A_3$, the last inequality reduces to

$$
\sum_{i=1}^{3} \sin A_i (\cos A_i - 2 \cos A_{i+1} A_{i+2}) \geq 0,
$$

which follows directly from the next Lemma.

**Lemma**  Let $ABC$ be a triangle. Then the **Cyclic Sum**

$$
\sum_{\text{cyc}} \sin A (\cos A - 2 \cos B \cos C) = 0.
$$

**Proof:** By the **double-angle formulas**, it suffices to show that

$$
\sum_{\text{cyc}} \sin 2A = 2 \sum_{\text{cyc}} \sin A \cos A = 4 \sum_{\text{cyc}} \sin A \cos B \cos C.
$$

Applying the **addition and subtraction formulas** gives

$$
\sin A \cos B \cos C + \sin B \cos C \cos A = \cos C (\sin A \cos B + \sin B \cos A) = \cos C \sin(A + B) = \cos C \sin C.
$$

Hence

$$
4 \sum_{\text{cyc}} \sin A \cos B \cos C
= 2 \sum_{\text{cyc}} (\sin A \cos B \cos C + \sin B \cos C \cos A)
= 2 \sum_{\text{cyc}} \cos C \sin C = \sum_{\text{cyc}} \sin 2C,
$$

as desired. ■

42. Let $ABC$ be a triangle. Let $x$, $y$, and $z$ be real numbers, and let $n$ be a positive integer. Prove the following four inequalities.

(a) [D. Barrow] $x^2 + y^2 + z^2 \geq 2yz \cos A + 2zx \cos B + 2xy \cos C$.

(b) [J. Wolstenholme]

$$
x^2 + y^2 + z^2 \geq 2(-1)^{n+1}(yz \cos nA + zx \cos nB + xy \cos nC).
$$
(c) [O. Bottema] $yza^2 + zxb^2 + xyc^2 \leq R^2(x + y + z)^2$.
(d) [A. Oppenheim] $xa^2 + yb^2 + zc^2 \geq 4[ABC] \sqrt{xy + yz + zx}$.

**Note:** These are very powerful inequalities, because $x, y, z$ can be arbitrary real numbers. By the same token, however, they are not easy to apply.

**Solution:** It is clear that part (a) is a special case of part (b) by setting $n = 1$. Parts (c) and (d) are two applications of part (b). Hence we prove only parts (b), (c), and (d).

(b) Rewrite the desired inequality as

$$x^2 + 2x (-1)^n(z \cos nB + y \cos nC) + y^2 + z^2 + 2(-1)^n yz \cos nA \geq 0.$$ 

Completing the square for $x^2 + 2x (-1)^n(z \cos nB + y \cos C)$ gives

$$\left[ x^2 + (-1)^n(z \cos nB + y \cos nC) \right]^2 + y^2 + z^2 + 2(-1)^n yz \cos nA \geq (z \cos nB + y \cos nC)^2$$

$$= z^2 \cos^2 nB + y^2 \cos^2 nC + 2yz \cos nB \cos nC.$$ 

It suffices to show that

$$y^2 + z^2 + 2(-1)^n yz \cos nA \geq z^2 \cos^2 nB + y^2 \cos^2 nC + 2yz \cos nB \cos nC,$$

or

$$y^2 \sin^2 nC + z^2 \sin^2 nB + 2yz \left[ (-1)^n \cos nA - \cos nB \cos nC \right] \geq 0.$$ (*)&

If $n = 2k$ is even, then $nA + nB + nC = 2k\pi$, and so $\cos nA = \cos (nB + nC) = \cos nB \cos nC - \sin nB \sin nC$. The desired inequality (*)& reduces to

$$y^2 \sin^2 nC + z^2 \sin^2 nB - 2yz \sin nB \sin nC = (y \sin nC - z \sin nB)^2 \geq 0,$$

which is evident.

If $n = 2k + 1$ is even, then $nA + nB + nC = (2k + 1)\pi$, and so $\cos nA = - \cos (nB + nC) = - \cos nB \cos nC + \sin nB \sin nC$. The desired inequality (*)& reduces to
\[ y^2 \sin^2 nC + z^2 \sin^2 nB - 2yz \sin nB \sin nC \]
\[ = (y \sin nC - z \sin nB)^2 \geq 0, \]

which is evident.

From the above proof, we note that the equality case of the desired inequality holds only if \((y \sin nC - z \sin nB)^2 \geq 0\), that is, if \(y \sin nC = z \sin nB\), or \(\frac{y}{\sin nB} = \frac{z}{\sin nC}\). By symmetry, the equality case holds only if \(\frac{x}{\sin nA} = \frac{y}{\sin nB} = \frac{z}{\sin nC}\).

It is also easy to check that the above condition is sufficient for the equality case to hold.

(c) The extended law of sines gives that \(\frac{a}{R} = 2 \sin A\) and its analogous forms for \(\frac{b}{R}\) and \(\frac{c}{R}\). Dividing both sides of the desired inequalities by \(R^2\) and expanding the resulting right-hand side yields

\[ 4(yz \sin^2 A + zx \sin^2 B + xy \sin^2 C) \leq x^2 + y^2 + z^2 + 2(xy + yz + zx), \]
or, by the double-angle formulas,

\[ x^2 + y^2 + z^2 \]
\[ \geq 2 \left[ yz(2 \sin^2 A - 1) + zx(2 \sin^2 B - 1) + xy(2 \sin^2 C - 1) \right] \]
\[ = -2(yz \cos 2A + zx \cos 2B + xy \cos 2C), \]

which is part (b) by setting \(n = 2\).

By the argument at the end of the proof of (b), we conclude that the equality case of the desired inequality holds if and only if

\[ \frac{x}{\sin 2A} = \frac{y}{\sin 2B} = \frac{z}{\sin 2C}. \]

(d) Setting \(x = xa^2\), \(y = yb^2\), and \(z = zc^2\) in part (c) gives

\[ a^2b^2c^2(xy + yz + zx) \leq R^2 \left( xa^2 + yb^2 + zc^2 \right)^2, \]
or

\[ 16R^2[ABC](xy + yz + zx) \leq R^2 \left( xa^2 + yb^2 + zc^2 \right)^2, \]

by Introductory Problem 25(a). Dividing both sides of the last inequality by \(R^2\) and taking square roots yields the desired result.
By the argument at the end of the proof of (b), we conclude that the equality case of the desired inequality holds if and only if
\[ \frac{xa^2}{\sin 2A} = \frac{yb^2}{\sin 2B} = \frac{zc^2}{\sin 2C}, \]
or
\[ \frac{xa}{\cos A} = \frac{yb}{\cos B} = \frac{zc}{\sin C}, \]
by the double-angle formulas and the law of sines. By the law of cosines, we have
\[ \frac{a}{\cos A} = \frac{2abc}{b^2 + c^2 - a^2} \]
and its analogous forms for \( \frac{b}{\cos B} \) and \( \frac{c}{\cos C} \). Therefore, the equality case holds if and only if
\[ \frac{x}{b^2 + c^2 - a^2} = \frac{y}{c^2 + a^2 - b^2} = \frac{z}{a^2 + b^2 - c^2}. \]

Note: The approach of completing the squares, shown in the proof of (b), is rather tricky. We can see this approach from another angle. Consider the quadratic function
\[ f(x) = x^2 - 2x(z \cos B + y \cos C) + y^2 + z^2 - 2yz \cos A. \]
Its discriminant is
\[ \Delta = 4(z \cos B + y \cos C)^2 - 4(y^2 + z^2 - 2yz \cos A) \]
\[ = 4 \left[ z^2 \cos^2 B - z^2 + 2yz(\cos A + \cos B \cos C) + y^2 \cos^2 C - y^2 \right] \]
\[ = 4 \left[ -z^2 \sin^2 B + 2yz(\cos(B+C) + \cos B \cos C) - y^2 \sin^2 C \right] \]
\[ = 4 \left[ -z^2 \sin^2 B + 2yz \sin B \sin C - y^2 \sin^2 C \right] \]
\[ = -4(z \sin B - y \sin C)^2 \leq 0. \]
For fixed \( y \) and \( z \) and large \( x \), \( f(x) \) is certainly positive. Hence \( f(x) \geq 0 \) for all \( x \), establishing (a). This method can be easily generalized to prove (b). We leave the generalization to the reader as an exercise.

43. [USAMO 2004] A circle \( \omega \) is inscribed in a quadrilateral \( ABCD \). Let \( I \) be the center of \( \omega \). Suppose that
Prove that $ABCD$ is an isosceles trapezoid.

**Note:** We introduce two trigonometric solutions and a synthetic-geometric solution. The first solution, by Oleg Golberg, is very technical. The second solution, by Tiankai Liu and Tony Zhang, reveals more geometrical background in their computations. This is by far the most challenging problem in the USAMO 2004. There were only four complete solutions. The fourth student is Jacob Tsimerman, from Canada. Evidently, these four students placed top four in the contest. There were nine IMO gold medals won by these four students, with each of Oleg and Tiankai winning three, Jacob two, and Tony one. Oleg won his first two representing Russia, and the third representing the United States. Jacob is one of only four students who achieved a perfect score at the IMO 2004 in Athens, Greece.

The key is to recognize that the given identity is a combination of equality cases of certain inequalities. By equal tangents, we have $|AB| + |CD| = |AD| + |BC|$ if and only if $ABCD$ has an incenter. We will prove that for a convex quadrilateral $ABCD$ with incenter $I$, then

$$(|AI| + |DI|)^2 + (|BI| + |CI|)^2 \leq (|AB| + |CD|)^2 = (|AD| + |BC|)^2.$$ (\*)

Equality holds if and only if $AD \parallel BC$ and $|AB| = |CD|$. Without loss of generality, we may assume that the inradius of $ABCD$ is 1.

**First Solution:** As shown in Figure 5.11, let $A_1, B_1, C_1,$ and $D_1$ be the points of tangency. Because circle $\omega$ is inscribed in $ABCD$, we can set $\angle D_1IA_1 = x, \angle A_1IB_1 = y, \angle B_1IC_1 = z, \angle C_1ID_1 = w$, and $x + y + z + w = 180^\circ$, or $x + w = 180^\circ - (y + z)$,
with \(0^\circ < x, y, z, w < 90^\circ\). Then \(|AI| = \sec x, |BI| = \sec y, |CI| = \sec z, |DI| = \sec w, |AD| = |AD_1| + |D_1D| = \tan x + \tan w\), and \(|BC| = |BB_1| + |B_1C| = \tan y + \tan z\). Inequality (\(\ast\)) becomes

\[
(\sec x + \sec w)^2 + (\sec y + \sec z)^2 \leq (\tan x + \tan y + \tan z + \tan w)^2.
\]

Expanding both sides of the above inequality and applying the identity \(\sec^2 x = 1 + \tan^2 x\) gives

\[
4 + 2(\sec x \sec w + \sec y \sec z) \leq 2 \tan x \tan y + 2 \tan x \tan z + 2 \tan x \tan w + 2 \tan y \tan z + 2 \tan y \tan w + 2 \tan z \tan w,
\]

or

\[
2 + \sec x \sec w + \sec y \sec z \leq \tan x \tan w + \tan y \tan z + (\tan x + \tan w)(\tan y + \tan z).
\]

Note that by the addition and subtraction formulas,

\[
1 - \tan x \tan w = \frac{\cos x \cos w - \sin x \sin w}{\cos x \cos w} = \frac{\cos(x + w)}{\cos x \cos w}.
\]

Hence,

\[
1 - \tan x \tan w + \sec x \sec w = \frac{1 + \cos(x + w)}{\cos x \cos w}.
\]

Similarly,

\[
1 - \tan y \tan z + \sec y \sec z = \frac{1 + \cos(y + z)}{\cos y \cos z}.
\]

Adding the last two equations gives

\[
2 + \sec x \sec w + \sec y \sec z - \tan x \tan w - \tan y \tan z = \frac{1 + \cos(x + w)}{\cos x \cos w} + \frac{1 + \cos(y + z)}{\cos y \cos z}.
\]

It suffices to show that

\[
\frac{1 + \cos(x + w)}{\cos x \cos w} + \frac{1 + \cos(y + z)}{\cos y \cos z} \leq (\tan x + \tan w)(\tan y + \tan z),
\]

or

\[
s + t \leq (\tan x + \tan w)(\tan y + \tan z),
\]

after setting \(s = \frac{1 + \cos(x + w)}{\cos x \cos w}\) and \(t = \frac{1 + \cos(y + z)}{\cos x \cos w}\). By the addition and subtraction formulas, we have

\[
\tan x + \tan w = \frac{\sin x \cos w + \cos x \sin w}{\cos x \cos w} = \frac{\sin(x + w)}{\cos x \cos w}.
\]
Similarly, 
\[ \tan y + \tan z = \frac{\sin(y + z)}{\cos y \cos z} = \frac{\sin(x + w)}{\cos y \cos z}, \]
because \( x + w = 180^\circ - (y + z) \). It follows that 
\[
(tan x + tan w)(tan y + tan z)
= \frac{\sin^2(x + w)}{\cos x \cos y \cos z \cos w} = \frac{1 - \cos^2(x + w)}{\cos x \cos y \cos z \cos w}
= \frac{[1 - \cos(x + w)][1 + \cos(x + w)]}{\cos x \cos y \cos z \cos w}
= \frac{[1 + \cos(y + z)][1 + \cos(x + w)]}{\cos x \cos y \cos z \cos w}
= st.
\]
The desired inequality becomes \( s+t \leq st \), or \((1-s)(1-t) = 1 - s - t + st \geq 1 \). It suffices to show that \( 1 - s \geq 1 \) and \( 1 - t \geq 1 \). By symmetry, we have only to show that \( 1 - s \geq 1 \); that is, 
\[
\frac{1 + \cos(x + w)}{\cos x \cos w} \geq 2.
\]
Multiplying both sides of the inequality by \( \cos x \cos w \) and applying the addition and subtraction formulas gives 
\[
1 + \cos x \cos w - \sin x \sin w \geq 2 \cos x \cos w,
\]
or \( 1 \geq \cos x \cos w + \sin x \sin w = \cos(x - w) \), which is evident. Equality holds if and only if \( x = w \). Therefore, inequality (*) is true, with equality if and only if \( x = w \) and \( y = z \), which happens precisely when \( AD \parallel BC \) and \( |AB| = |CD| \), as was to be shown.

**Second Solution:** We maintain the same notation as in the first solution. Applying the law of cosines to triangles \( ADI \) and \( BCI \) gives 
\[
|AI|^2 + |DI|^2 = 2 \cos(x + w)|AI| \cdot |DI| + |AD|^2
\]
and 
\[
|BI|^2 + |CI|^2 = 2 \cos(y + z)|BI| \cdot |CI| + |BC|^2.
\]
Adding the last two equations and completing squares gives 
\[
(|AI| + |DI|)^2 + (|BI| + |CI|)^2 + 2|AD| \cdot |BC|
= 2 \cos(x + w)|AI| \cdot |DI| + 2 \cos(y + z)|BI| \cdot |CI|
+ 2|AI| \cdot |DI| + 2|BI| \cdot |CI| + (|AD| + |BC|)^2.
\]
Hence, establishing inequality (*) is equivalent to establishing the inequality

\[ [1 + \cos(x + w)]|AI| \cdot |DI| + [1 + \cos(y + z)]|BI| \cdot |CI| \leq |AD| \cdot |BC|. \]

Since \(2|ADI| = |AD| \cdot |ID| = |AI| \cdot |DI| \sin(x+w), |AI| \cdot |DI| = \frac{|AD|}{\sin(x+w)}\).

Similarly, \(|BI| \cdot |CI| = \frac{|BC|}{\sin(y+z)}\). Because \(x + w = 180^\circ - (y + z)\), we have

\[\sin(x + w) = \sin(y + z)\text{ and }\cos(x + w) = -\cos(x + w).\]

Plugging all the above information back into the last inequality yields

\[\frac{1 + \cos(x + w)}{\sin(x + w)} \cdot |AD| + \frac{1 - \cos(x + w)}{\sin(x + w)} \cdot |BC| \leq |AD| \cdot |BC|,\]

or

\[\frac{1 + \cos(x + w)}{|BC|} + \frac{1 - \cos(x + w)}{|AD|} \leq \sin(x + w). \quad (**)\]

Note that by the addition and subtraction, the product-to-sum, and the double-angle formulas, we have

\[|AD| = |AD_1| + |D_1D| = \tan x + \tan w = \frac{\sin x}{\cos x} + \frac{\sin w}{\cos w} = \frac{\sin(x + w)}{\cos x \cos w} = \frac{2\sin(x + w)}{2 \cos x \cos w} = \frac{4 \sin \frac{x + w}{2} \cos \frac{x + w}{2}}{\cos(x + w) + 1} \geq \frac{4 \sin \frac{x + w}{2} \cos \frac{x + w}{2}}{2 \cos^2 \frac{x + w}{2}} = 2 \tan \frac{x + w}{2}.\]

Equality holds if and only if \(\cos(x - w) = 1\), that is, if \(x = w\). (This step can be done easily by applying Jensen’s inequality, using the fact \(y = \tan x\) is convex for \(0^\circ < x < 90^\circ\).) Consequently, by the double-angle formulas,

\[\frac{1 - \cos(x + w)}{|AD|} \leq \frac{2 \sin^2 \frac{x + w}{2}}{2 \tan \frac{x + w}{2}} = \frac{\sin \frac{x + w}{2} \cos \frac{x + w}{2}}{\sin(x + w)} = \frac{2 \sin \frac{x + w}{2}}{2 \tan \frac{x + w}{2}} = \frac{\sin(x + w)}{2}.\]

In exactly the same way, we can show that

\[\frac{1 + \cos(x + w)}{|BC|} = \frac{1 - \sin(y + z)}{|BC|} \leq \frac{\sin(y + z)}{2} = \frac{\sin(x + w)}{2}.\]

Adding the last two inequalities gives the desired inequality (**). Equality holds if and only if \(x = w\) and \(y = z\), which happens precisely when \(AD \parallel BC\) and \(|AB| = |CD|\), as was to be shown.
Third Solution: Because the circle $\omega$ is inscribed in $ABCD$, as shown in Figure 5.12, we can set $\angle DAI = \angle IAB = a$, $\angle ABI = \angle IBC = b$, $\angle BCI = \angle ICD = c$, $\angle CDAI = \angle IDA = d$, and $a + b + c + d = 180^\circ$. Our proof is based on the following key Lemma.

Lemma: If a circle $\omega$, centered at $I$, is inscribed in a quadrilateral $ABCD$, then

$$|BI|^2 + \frac{|AI|}{|DI|} \cdot |BI| \cdot |CI| = |AB| \cdot |BC|. \quad (\dagger)$$

Proof: Construct a point $P$ outside of the quadrilateral such that triangle $ABP$ is similar to triangle $DCI$. We obtain

$$\angle PAI + \angle PBI = \angle PAB + \angle BAI + \angle PBA + \angle ABI$$

$$= \angle IDC + a + \angle ICD + b$$

$$= a + b + c + d = 180^\circ,$$

implying that the quadrilateral $PAIB$ is cyclic. By Ptolemy’s theorem, we have $|AI| \cdot |BP| + |BI| \cdot |AP| = |AB| \cdot |IP|$, or

$$|BP| \cdot \frac{|AI|}{|IP|} + |BI| \cdot \frac{|AP|}{|IP|} = |AB|. \quad (\dagger\dagger)$$

Because $PAIB$ is cyclic, it is not difficult to see that, as indicated in the figure, $\angle IPB = \angle IAB = a$, $\angle API = \angle ABI = b$, $\angle AIP = \angle ABP = c$, and $\angle IPB = \angle PAB = d$. Note that triangles $AIP$ and $ICB$ are similar, implying that

$$\frac{|AI|}{|IP|} = \frac{|IC|}{|CB|} \quad \text{and} \quad \frac{|AP|}{|IP|} = \frac{|IB|}{|CB|}.$$

Substituting the above equalities into the identity $(\dagger\dagger)$, we arrive at

$$|BP| \cdot \frac{|CI|}{|BC|} + \frac{|BI|^2}{|BC|} = |AB|.$$
or

\[ |BP| \cdot |CI| + |BI|^2 = |AB| \cdot |BC|. \]  \(\dagger\dagger\dagger\)

Note also that triangle \(BIP\) and triangle \(IDA\) are similar, implying that \(\frac{|BP|}{|BI|} = \frac{|IA|}{|ID|}\), or

\[ |BP| = \frac{|AI|}{|ID|} \cdot |IB|. \]

Substituting the above identity back into \(\dagger\dagger\dagger\) gives the desired relation \(\dagger\), establishing the Lemma.

Now we prove our main result. By the Lemma and symmetry, we have

\[ |CI|^2 + \frac{|DI|}{|AI|} \cdot |BI| \cdot |CI| = |CD| \cdot |BC|. \] \(\ddagger\)

Adding the two identities \(\dagger\) and \(\ddagger\) gives

\[ |BI|^2 + |CI|^2 + \left(\frac{|AI|}{|DI|} + \frac{|DI|}{|AI|}\right)|BI| \cdot |CI| = |BC|(|AB| + |CD|). \]

By the arithmetic–geometric means inequality, we have \(\frac{|AI|}{|DI|} + \frac{|DI|}{|AI|} \geq 2\). Thus,

\[ |BC|(|AB| + |CD|) \geq |IB|^2 + |IC|^2 + 2|IB| \cdot |IC| = (|BI| + |CI|)^2, \]

where equality holds if and only if \(|AI| = |DI|\). Likewise, we have

\[ |AD|(|AB| + |CD|) \geq (|AI| + |DI|)^2, \]

where equality holds if and only if \(|BI| = |CI|\). Adding the last two identities gives the desired inequality \((\ast\ast)\).

By the given condition in the problem, all the equalities in the above discussion must hold; that is, \(|AI| = |DI|\) and \(|BI| = |CI|\). Consequently, we have \(a = d, b = c\), and so \(\angle DAB + \angle ABC = 2a + 2b = 180^\circ\), implying that \(AD \parallel BC\). It is not difficult to see that triangle \(AIB\) and triangle \(DIC\) are congruent, implying that \(|AB| = |CD|\). Thus, \(ABCD\) is an isosceles trapezoid.

44. [USAMO 2001] Let \(a, b, \) and \(c\) be nonnegative real numbers such that

\[ a^2 + b^2 + c^2 + abc = 4. \]

Prove that

\[ 0 \leq ab + bc + ca - abc \leq 2. \]
The proof of the lower bound is rather simple. From the given condition, at least one of \(a, b, \) and \(c\) does not exceed 1, say \(a \leq 1\). Then

\[
ab + bc + ca - abc = a(b + c) + bc(1 - a) \geq 0.
\]

To obtain equality, we have \(a(b + c) = bc(1 - a) = 0\). If \(a = 1\), then \(b + c = 0\) or \(b = c = 0\), which contradicts the fact that \(a^2 + b^2 + c^2 + abc = 4\). Hence \(1 - a \neq 0\), and only one of \(b\) and \(c\) is 0. Without loss of generality, say \(b = 0\). Therefore \(b + c > 0\) and \(a = 0\). Plugging \(a = b = 0\) back into the given condition, we get \(c = 2\). By permutation, the lower bound holds if and only if \((a, b, c)\) is one of the triples \((2, 0, 0)\), \((0, 2, 0)\), and \((0, 0, 2)\). We next present three proofs of the upper bound.

**First Solution:** Based on Introductory Problem 22, we set \(a = 2 \sin \frac{A}{2}\), \(b = 2 \sin \frac{B}{2}\), and \(c = 2 \sin \frac{C}{2}\), where \(ABC\) is a triangle. We have

\[
ab = 4 \sin \frac{A}{2} \sin \frac{B}{2} = 2 \sqrt{\sin A \tan \frac{A}{2} \sin B \tan \frac{B}{2}}
\]

\[
= 2 \sqrt{\sin A \tan \frac{B}{2} \sin B \tan \frac{A}{2}}.
\]

By the **arithmetic–geometric means inequality**, this is at most

\[
\sin A \tan \frac{B}{2} + \sin B \tan \frac{A}{2} = \sin A \cot \frac{A + C}{2} + \sin B \cot \frac{B + C}{2}.
\]

Likewise,

\[
bc \leq \sin B \cot \frac{B + A}{2} + \sin C \cot \frac{C + A}{2},
\]

\[
ca \leq \sin C \cot \frac{C + B}{2} + \sin A \cot \frac{A + B}{2}.
\]

Therefore, by the **sum-to-product**, **product-to-sum**, and the **double-angle formulas**, we have
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\[ ab + bc + ca \]
\[ \leq (\sin A + \sin B) \cot \frac{A + B}{2} + (\sin B + \sin C) \cot \frac{B + C}{2} \]
\[ + \quad (\sin C + \sin A) \cot \frac{C + A}{2} \]
\[ = 2 \cos \frac{A - B}{2} \cos \frac{A + B}{2} + 2 \cos \frac{B - C}{2} \cos \frac{B + C}{2} \]
\[ + \quad 2 \cos \frac{C - A}{2} \cos \frac{C + A}{2} \]
\[ = 2(\cos A + \cos B + \cos C) \]
\[ = 6 - 4 \left( \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \right) \]
\[ = 6 - \left( a^2 + b^2 + c^2 \right). \]

Using the given equality, this last quantity equals \( 2 + abc \). It follows that
\[ ab + bc + ca \leq 2 + abc, \]
as desired.

**Second Solution:** Clearly, \( 0 \leq a, b, c \leq 2 \). In the light of Introductory Problem 24(d), we can set \( a = 2 \cos A \), \( b = 2 \cos B \), and \( c = 2 \cos C \), where \( ABC \) is an acute triangle. Either two of \( A, B \), and \( C \) are at least \( 60^\circ \), or two of \( A, B \), and \( C \) are at most \( 60^\circ \). Without loss of generality, assume that \( A \) and \( B \) have this property.

With these trigonometric substitutions, we find that the desired inequality is equivalent to
\[ 2(\cos A \cos B + \cos B \cos C + \cos C \cos A) \leq 1 + 4 \cos A \cos B \cos C, \]
or,
\[ 2(\cos A \cos B + \cos B \cos C + \cos C \cos A) \]
\[ \leq 3 - 2(\cos^2 A + \cos^2 B + \cos^2 C). \]

Hence, by the **double-angle formulas,** it suffices to prove that
\[ \cos 2A + \cos 2B + \cos 2C \]
\[ + 2(\cos A \cos B + 2 \cos B \cos C + \cos C \cos A) \leq 0. \]
By the sum-to-product and double-angle formulas, the sum of the first three terms in this inequality is

\[
\cos 2A + \cos 2B + \cos 2C \\
= 2 \cos (A + B) \cos (A - B) + 2 \cos^2 (A + B) - 1 \\
= 2 \cos (A + B)[\cos (A - B) + \cos (A + B)] - 1 \\
= 4 \cos (A + B) \cos A \cos B - 1,
\]

while the remaining terms equal

\[
2 \cos A \cos B + 2 \cos C(\cos A + \cos B) \\
= \cos (A + B) + \cos (A - B) - 2 \cos (A + B)(\cos A + \cos B),
\]

by the product-to-sum formulas. Hence, it suffices to prove that

\[
\cos (A + B)[4 \cos A \cos B + 1 - 2 \cos A - 2 \cos B] + \cos (A - B) \leq 1,
\]
or

\[
- \cos C(1 - 2 \cos A)(1 - 2 \cos B) + \cos (A - B) \leq 1. \quad (\ast)
\]

We consider the following cases:

(i) *At least one angle is 60°.* If \( A \) or \( B \) equals 60°, then we may assume, without loss of generality, that \( A = 60° \). If \( C = 60° \), then because either \( A, B \geq 60° \) or \( A, B \leq 60° \), we must actually have \( A = B = 60° \), in which case equality holds. In either case, we may assume \( A = 60° \). Then \( (\ast) \) becomes \( \cos (A - B) \leq 1 \), which is always true, and equality holds if and only if \( A = B = C = 60° \), that is, if and only if \( a = b = c = 1 \).

(ii) *No angle equals 60°.* Because either \( A, B \geq 60° \) or \( A, B \leq 60° \), we have \((1 - 2 \cos A)(1 - 2 \cos B) > 0 \). Since \( \cos C \geq 0 \) and \( \cos (A - B) \leq 1 \), \( (\ast) \) is always true. Equality holds when \( \cos C = 0 \) and \( \cos (A - B) = 1 \). This holds exactly when \( A = B = 45° \) and \( C = 90° \); that is, when \( a = b = \sqrt{2} \) and \( c = 0 \).

**Third Solution:** The problem also admits the following clever purely algebraic method, which is due to Oaz Nir and Richard Stong, independently.

Either two of \( a, b, c \) are less than or equal to 1, or two are greater than or equal to 1. Assume that \( b \) and \( c \) have this property. Then

\[
b + c - bc = 1 - (1 - b)(1 - c) \leq 1. \quad (\dagger)
\]
Viewing the given equality as a quadratic equation in \(a\) and solving for \(a\) yields

\[
a = \frac{-bc \pm \sqrt{b^2c^2 - 4(b^2 + c^2) + 16}}{2}.
\]

Note that

\[
b^2c^2 - 4(b^2 + c^2) + 16 \leq b^2c^2 - 8bc + 16 = (4 - bc)^2.
\]

For the given equality to hold, we must have \(b, c \leq 2\), so that \(4 - bc \geq 0\). Hence,

\[
a \leq \frac{-bc + |4 - bc|}{2} = \frac{-bc + 4 - bc}{2} = 2 - bc,
\]

or

\[
2 - bc \geq a. \quad (\ddagger)
\]

Combining the inequalities (\(\dagger\)) and (\(\ddagger\)) gives

\[
2 - bc = (2 - bc) \cdot 1 \geq a(b + c - bc) = ab + ac - abc,
\]

or \(ab + ac + bc - abc \leq 2\), as desired.

45. [Gabriel Dospinescu and Dung Tran Nam] Let \(s, t, u, v\) be numbers in the interval \((0, \frac{\pi}{2})\) with \(s + t + u + v = \pi\). Prove that

\[
\frac{\sqrt{2} \sin s - 1}{\cos s} + \frac{\sqrt{2} \sin t - 1}{\cos t} + \frac{\sqrt{2} \sin u - 1}{\cos u} + \frac{\sqrt{2} \sin v - 1}{\cos v} \geq 0.
\]

**Solution:** Set \(a = \tan s, b = \tan t, c = \tan u,\) and \(d = \tan v\). Then \(a, b, c, d\) are positive real numbers. Because \(s + t + u + v = \pi\), it follows that \(\tan(s + t) + \tan(u + v) = 0\); that is,

\[
\frac{a + b}{1 - ab} + \frac{c + d}{1 - cd} = 0,
\]

by the **addition and subtraction formulas**. Multiplying both sides of the last equation by \((1 - ab)(1 - cd)\) yields

\[
(a + b)(1 - cd) + (c + d)(1 - ab) = 0,
\]

or

\[
a + b + c + d = abc + bcd + cda + dab.
\]
Consequently, we obtain

\[(a + b)(a + c)(a + d) = a^2(a + b + c + d) + abc +bcd + cda + dab\]

\[= (a^2 + 1)(a + b + c + d),\]

or

\[\frac{a^2 + 1}{a + b} = \frac{(a + c)(a + d)}{a + b + c + d}\]

and its analogous forms. Hence

\[
\frac{a^2 + 1}{a + b} + \frac{b^2 + 1}{b + c} + \frac{c^2 + 1}{c + d} + \frac{d^2 + 1}{d + a} = \frac{(a + c)(a + d) + (b + d)(b + a) + (c + a)(c + b) + (d + b)(d + c)}{a + b + c + d}
\]

\[= a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)
\]

\[= a + b + c + d.
\]

By Cauchy–Schwarz inequality, we have

\[2(a + b + c + d)^2
\]

\[= 2(a + b + c + d) \left( \frac{a^2 + 1}{a + b} + \frac{b^2 + 1}{b + c} + \frac{c^2 + 1}{c + d} + \frac{d^2 + 1}{d + a} \right)
\]

\[= \left[ (a + b) + (b + c) + (c + d) + (d + a) \right]
\]

\[\times \left( \frac{a^2 + 1}{a + b} + \frac{b^2 + 1}{b + c} + \frac{c^2 + 1}{c + d} + \frac{d^2 + 1}{d + a} \right)
\]

\[\geq \left( \sqrt{a^2 + 1} + \sqrt{b^2 + 1} + \sqrt{c^2 + 1} + \sqrt{d^2 + 1} \right)^2,
\]

or

\[\sqrt{a^2 + 1} + \sqrt{b^2 + 1} + \sqrt{c^2 + 1} + \sqrt{d^2 + 1} \leq \sqrt{2}(a + b + c + d).
\]

The least inequality is equivalent to

\[\frac{1}{\cos s} + \frac{1}{\cos t} + \frac{1}{\cos u} + \frac{1}{\cos v} \leq \sqrt{2} \left( \frac{\sin s}{\cos s} + \frac{\sin t}{\cos t} + \frac{\sin u}{\cos u} + \frac{\sin v}{\cos v} \right),
\]

from which the desired inequality follows.
46. [USAMO 1995] Suppose a calculator is broken and the only keys that still work are the sin, cos, tan, sin\(^{-1}\), cos\(^{-1}\), and tan\(^{-1}\) buttons. The display initially shows 0. Given any positive rational number \(q\), show that we can get \(q\) to appear on the display panel of the calculator by pressing some finite sequence of buttons. Assume that the calculator does real-number calculations with infinite precision, and that all functions are in terms of radians.

Solution: Because \(\cos^{-1}\sin \theta = \frac{\pi}{2} - \theta\) and \(\tan \left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan \theta}\) for \(0 < \theta < \frac{\pi}{2}\), we have for any \(x > 0\),

\[
\tan \cos^{-1} \sin^{-1} x = \tan \left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{1}{x}.
\] (*)

Also, for \(x \geq 0\),

\[
\cos \tan^{-1} \sqrt{x} = \frac{1}{\sqrt{x} + 1},
\]
so by (*),

\[
\tan \cos^{-1} \sin^{-1} \cos \tan^{-1} \sqrt{x} = \sqrt{x + 1}.
\] (**)

By induction on the denominator of \(r\), we now prove that \(\sqrt{r}\), for every non-negative rational number \(r\), can be obtained by using the operations

\[
\sqrt{x} \mapsto \sqrt{x + 1} \quad \text{and} \quad x \mapsto \frac{1}{x}.
\]

If the denominator is 1, we can obtain \(\sqrt{r}\), for every nonnegative integer \(r\), by repeated application of \(\sqrt{x} \mapsto \sqrt{x + 1}\). Now assume that we can get \(\sqrt{r}\) for all rational numbers \(r\) with denominator up to \(n\). In particular, we can get any of

\[
\sqrt{\frac{n+1}{1}, \sqrt{\frac{n+1}{2}, \ldots, \sqrt{\frac{n+1}{n}}},}
\]
so we can also get

\[
\sqrt{\frac{1}{n+1}, \sqrt{\frac{2}{n+1}, \ldots, \sqrt{\frac{n}{n+1}}},}
\]
and \(\sqrt{r}\), for any positive \(r\) of exact denominator \(n + 1\), can be obtained by repeatedly applying \(\sqrt{x} \mapsto \sqrt{x + 1}\).

Thus for any positive rational number \(r\), we can obtain \(\sqrt{r}\). In particular, we can obtain \(\sqrt{q^2} = q\).
47. [China 2003, by Yumin Huang] Let \( n \) be a fixed positive integer. Determine the smallest positive real number \( \lambda \) such that for any \( \theta_1, \theta_2, \ldots, \theta_n \) in the interval \((0, \frac{\pi}{2})\), if

\[
\tan \theta_1 \tan \theta_2 \cdots \tan \theta_n = 2^{n/2},
\]

then

\[
\cos \theta_1 + \cos \theta_2 + \cdots + \cos \theta_n \leq \lambda.
\]

**Solution:** The answer is

\[
\lambda = \begin{cases} 
\frac{\sqrt{3}}{3}, & n = 1; \\
\frac{2\sqrt{3}}{3}, & n = 2; \\
 n - 1, & n \geq 3.
\end{cases}
\]

The case \( n = 1 \) is trivial. If \( n = 2 \), we claim that

\[
\cos \theta_1 + \cos \theta_2 \leq \frac{2\sqrt{3}}{3},
\]

with equality if and only if \( \theta_1 = \theta_2 = \tan^{-1} \sqrt{2} \). It suffices to show that

\[
\cos^2 \theta_1 + \cos^2 \theta_2 + 2 \cos \theta_1 \cos \theta_2 \leq \frac{4}{3},
\]

or

\[
\frac{1}{1 + \tan^2 \theta_1} + \frac{1}{1 + \tan^2 \theta_2} + 2 \sqrt{\frac{1}{(1 + \tan^2 \theta_1)(1 + \tan^2 \theta_2)}} \leq \frac{4}{3}.
\]

Because \( \tan \theta_1 \tan \theta_2 = 2 \),

\[
(1 + \tan^2 \theta_1)(1 + \tan^2 \theta_2) = 5 + \tan^2 \theta_1 + \tan^2 \theta_2.
\]

By setting \( \tan^2 \theta_1 + \tan^2 \theta_2 = x \), the last inequality becomes

\[
\frac{2 + x}{5 + x} + 2 \sqrt{\frac{1}{5 + x}} \leq \frac{4}{3},
\]

or

\[
2 \sqrt{\frac{1}{5 + x}} \leq \frac{14 + x}{3(5 + x)}.
\]

Squaring both sides and clearing denominators, we get \( 36(5 + x) \leq 196 + 28x + x^2 \), that is, \( 0 \leq x^2 - 8x + 16 = (x - 4)^2 \). This establishes our claim.
Now assume that \( n \geq 3 \). We claim that \( \lambda = n - 1 \). Note that \( \lambda \geq n - 1 \); by setting \( \theta_2 = \theta_3 = \cdots = \theta_n = \theta \) and letting \( \theta \to 0 \), we find that \( \theta_1 \to \frac{\pi}{2} \), and so the left-hand side of the desired inequality approaches \( n - 1 \). It suffices to show that

\[
\cos \theta_1 + \cos \theta_2 + \cdots + \cos \theta_n \leq n - 1.
\]

Without loss of generality, assume that \( \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n \). Then

\[
\tan \theta_1 \tan \theta_2 \tan \theta_3 \geq 2 \sqrt{2}.
\]

It suffices to show that

\[
\cos \theta_1 + \cos \theta_2 + \cos \theta_3 < 2.
\]

Because \( \sqrt{1 - x^2} \leq 1 - \frac{1}{2}x^2 \), \( \cos \theta_i = \sqrt{1 - \sin^2 \theta_i} < 1 - \frac{1}{2} \sin^2 \theta_i \). Consequently, by the arithmetic–geometric means inequality,

\[
\cos \theta_2 + \cos \theta_3 < 2 - \frac{1}{2} \left( \sin^2 \theta_2 + \sin^2 \theta_3 \right) \leq 2 - \sin \theta_2 \sin \theta_3.
\]

Because

\[
\tan^2 \theta_1 \geq \frac{8}{\tan^2 \theta_2 \tan^2 \theta_3},
\]

we have

\[
\sec^2 \theta_1 \geq \frac{8 + \tan^2 \theta_2 \tan^2 \theta_3}{\tan^2 \theta_2 \tan^2 \theta_3},
\]

or

\[
\cos \theta_1 \leq \frac{\tan \theta_2 \tan \theta_3}{\sqrt{8 + \tan^2 \theta_2 \tan^2 \theta_3}} = \frac{\sin \theta_2 \sin \theta_3}{\sqrt{8 \cos^2 \theta_2 \cos^2 \theta_3 + \sin^2 \theta_2 \sin^2 \theta_3}}.
\]

It follows that

\[
\cos \theta_1 + \cos \theta_2 + \cos \theta_3
\]

\[
< 2 - \sin \theta_2 \sin \theta_3 \left[ 1 - \frac{1}{8 \cos^2 \theta_2 \cos^2 \theta_3 + \sin^2 \theta_2 \sin^2 \theta_3} \right].
\]

It is clear that the equality

\[
8 \cos^2 \theta_2 \cos^2 \theta_3 + \sin^2 \theta_2 \sin^2 \theta_3 \geq 1,
\]

establishes the desired inequality \((*)\). Inequality \((**)\) is equivalent to

\[
8 + \tan^2 \theta_2 \tan^2 \theta_3 \geq (1 + \tan^2 \theta_2)(1 + \tan^2 \theta_3),
\]

\[
\tan^2 \theta_1 \geq \frac{8}{\tan^2 \theta_2 \tan^2 \theta_3},
\]

we have

\[
\sec^2 \theta_1 \geq \frac{8 + \tan^2 \theta_2 \tan^2 \theta_3}{\tan^2 \theta_2 \tan^2 \theta_3},
\]

or

\[
\cos \theta_1 \leq \frac{\tan \theta_2 \tan \theta_3}{\sqrt{8 + \tan^2 \theta_2 \tan^2 \theta_3}} = \frac{\sin \theta_2 \sin \theta_3}{\sqrt{8 \cos^2 \theta_2 \cos^2 \theta_3 + \sin^2 \theta_2 \sin^2 \theta_3}}.
\]

It follows that

\[
\cos \theta_1 + \cos \theta_2 + \cos \theta_3
\]

\[
< 2 - \sin \theta_2 \sin \theta_3 \left[ 1 - \frac{1}{8 \cos^2 \theta_2 \cos^2 \theta_3 + \sin^2 \theta_2 \sin^2 \theta_3} \right].
\]

It is clear that the equality

\[
8 \cos^2 \theta_2 \cos^2 \theta_3 + \sin^2 \theta_2 \sin^2 \theta_3 \geq 1,
\]

establishes the desired inequality \((*)\). Inequality \((**)\) is equivalent to

\[
8 + \tan^2 \theta_2 \tan^2 \theta_3 \geq (1 + \tan^2 \theta_2)(1 + \tan^2 \theta_3),
\]
or
\[ 7 \geq \tan^2 \theta_2 + \tan^2 \theta_3. \]
Thus if \( \tan^2 \theta_2 + \tan^2 \theta_3 \leq 7 \), then inequality (*) holds, and we are done.
Assume that \( \tan^2 \theta_2 + \tan^2 \theta_3 > 7 \). Then \( \tan^2 \theta_1 \geq \tan^2 \theta_2 \geq \frac{7}{2} \). Then
\[
\cos \theta_1 \leq \cos \theta_2 = \frac{1}{\sqrt{1 + \tan^2 \theta_2}} \leq \frac{\sqrt{2}}{3},
\]
implying that
\[
\cos \theta_1 + \cos \theta_2 + \cos \theta_3 \leq \frac{2\sqrt{3}}{3} + 1 < 2,
\]
establishing (*) again.
Therefore, inequality (*) is true, as desired.

48. Let \( ABC \) be an acute triangle. Prove that
\[
(sin 2B + sin 2C)^2 \sin A + (sin 2C + sin 2A)^2 \sin B + (sin 2A + sin 2B)^2 \sin C \leq 12 \sin A \sin B \sin C.
\]

**First Solution:** Applying the addition and subtraction formulas gives
\[
(sin 2B + sin 2C)^2 \sin A = 4 \sin^2(B + C) \cos^2(B - C) \sin A
= 4 \sin^3 A \cos^2(B - C),
\]
because \( A + B + C = 180^\circ \). Hence it suffices to show that the cyclic sum
\[
\sum_{cyc} \sin^3 A \cos^2(B - C)
\]
is less than or equal to \( 3 \sin A \sin B \sin C \), which follows from
\[
\sum_{cyc} 4 \sin^3 A \cos(B - C) = 12 \sin A \sin B \sin C.
\]
Indeed, we have
\[
4 \sin^3 A \cos(B - C)
= 4 \sin^2 A \sin(B + C) \cos(B - C)
= 2 \sin^2 A (\sin 2B + \sin 2C)
= (1 - \cos 2A)(\sin 2B + \sin 2C)
= (\sin 2B + \sin 2C) - \sin 2B \cos 2A - \sin 2C \cos 2A.
\]
It follows that

\[
\sum_{\text{cyc}} 4 \sin^3 A \cos(B - C)
\]

\[= \sum_{\text{cyc}} (\sin 2B + \sin 2C) - \sum_{\text{cyc}} \sin 2B \cos 2A - \sum_{\text{cyc}} \sin 2C \cos 2A
\]

\[= 2 \sum_{\text{cyc}} \sin 2A - \sum_{\text{cyc}} \sin 2B \cos 2A - \sum_{\text{cyc}} \sin 2A \cos 2B
\]

\[= 2 \sum_{\text{cyc}} \sin 2A - \sum_{\text{cyc}} (\sin 2B \cos 2A + \sin 2A \cos 2B)
\]

\[= 2 \sum_{\text{cyc}} \sin 2A - \sum_{\text{cyc}} (2B + 2A)
\]

\[= 2 \sum_{\text{cyc}} \sin 2A + \sum_{\text{cyc}} \sin 2C
\]

\[= 3(\sin 2A + \sin 2B + \sin 2C)
\]

\[= 12 \sin A \sin B \sin C,
\]

by Introductory Problem 24(a). Equality holds if and only if \(\cos(A - B) = \cos(B - C) = \cos(C - A) = 1\), that is, if and only if triangle \(ABC\) is equilateral.

**Note:** Enlarging \(\sin^3 A \cos^2(B - C)\) to \(\sin^3 A \cos(B - C)\) is a very clever but somewhat tricky idea. The following more geometric approach reveals more of the motivation behind the problem. Please note the last step in the proof of the Lemma below.

**Second Solution:** We can rewrite the desired inequality as

\[
\sum_{\text{cyc}} (\sin 2B + \sin 2C)^2 \sin A \leq 12 \sin A \sin B \sin C.
\]

By the extended law of sines, we have \(c = 2R \sin C\), \(a = 2R \sin A\), and \(b = 2R \sin B\). Hence

\[12R^2 \sin A \sin B \sin C = 3ab \sin C = 6[ABC].
\]

It suffices to show that

\[R^2 \sum_{\text{cyc}} (\sin 2B + \sin 2C)^2 \sin A \leq 6[ABC]. \quad (*)
\]
We establish the following Lemma.

**Lemma**  Let $AD, BE, CF$ be the altitudes of acute triangle $ABC$, with $D, E, F$ on sides $BC, CA, AB$, respectively. Then

$$|DE| + |DF| \leq |BC|.$$  

Equality holds if and only if $|AB| = |AC|$.

**Proof:** We consider Figure 5.13. Because $\angle CFA = \angle CDA = 90^\circ$, quadrilateral $AFDC$ is cyclic, and so $\angle FDB = \angle BAC = \angle CAB$ and $\angle BFD = \angle BCA = \angle BCA$. Hence triangles $BDF$ and $BAC$ are similar, so

$$\frac{|DF|}{|AC|} = \frac{|BF|}{|BC|} = \cos B,$$

or (by the double-angle formula)

$$|DF| = b \cos B = 2R \sin B \cos B = R \sin 2B.$$  

Likewise, $|DE| = c \cos C = R \sin 2C$. Thus,

$$|DE| + |DF| = R(\sin 2B + \sin 2C). \quad (\dagger)$$

Since $0^\circ < A, B, C < 90^\circ$, by the sum-to-product formula,

$$|BC| - (|DE| + |DF|) = R[2 \sin A - (\sin 2B + \sin 2C]$$

$$= R[2 \sin A - 2 \sin (B + C) \cos (B - C)]$$

$$= 2R \sin A[1 - \cos (B - C)] \geq 0,$$

as desired. (This can also be proven by the law of cosines.)
5. Solutions to Advanced Problems

Because both $ABDE$ and $ACDF$ are cyclic, $\angle BDF = \angle CDE = \angle CAB$. Thus, by the Lemma, we have

$$2([BFC] + [BEC]) = |DF| \cdot |BC| \cdot \sin \angle BDF + |DE| \cdot |BC| \cdot \sin \angle EDC$$
$$= |BC|(|DE| + |DF|) \sin \angle A \geq (|DE| + |DF|)^2 \sin \angle A.$$

By equation (†), the last inequality is equivalent to

$$R^2(\sin 2B + \sin 2C)^2 \sin A \leq 2[BFC] + 2[BEC].$$
Likewise, we have

$$R^2(\sin 2C + \sin 2A)^2 \sin B \leq 2[CDA] + 2[CFA]$$

and

$$R^2(\sin 2A + \sin 2B)^2 \sin C \leq 2[AEB] + 2[ADB].$$
Adding the last three inequalities yields the desired result. In view of the Lemma, it is also clear that equality holds if and only if triangle $ABC$ is equilateral.

49. [Bulgaria 1998] On the sides of a nonobtuse triangle $ABC$ are constructed externally a square $P_4$, a regular $m$-sided polygon $P_m$, and a regular $n$-sided polygon $P_n$. The centers of the square and the two polygons form an equilateral triangle. Prove that $m = n = 6$, and find the angles of triangle $ABC$.

Solution: The angles are $90^\circ$, $45^\circ$, and $45^\circ$. We prove the following lemma.

**Lemma** Let $O$ be a point inside equilateral triangle $XYZ$. If

$$\angle YOZ = x, \quad \angle ZOX = y, \quad \angle XOY = z.$$

then

$$\frac{|OX|}{\sin(x - 60^\circ)} = \frac{|OY|}{\sin(y - 60^\circ)} = \frac{|OZ|}{\sin(z - 60^\circ)}.$$

Proof: As shown in Figure 5.14, let $R$ denote clockwise rotation of $60^\circ$ around the point $Z$, and let $R(X) = X_1$ and $R(O) = O_1$. 

Then \( R(Y) = X \), and so triangle \( ZO_1O \) is equilateral. Consequently, triangles \( ZO_1X \) and \( ZOY \) are congruent, and so \( |O_1X| = |OY| \). Note that \( x + y + z = 360^\circ \). We have

\[
\begin{align*}
\angle O_1OX &= \angle ZOX - \angle ZO_1O = \angle ZOX - 60^\circ = y - 60^\circ, \\
\angle XO_1O &= \angle XO_1Z - \angle O_1OZ = \angle YOZ - 60^\circ = x - 60^\circ, \\
\angle OXO_1 &= 180^\circ - \angle O_1OX - \angle XO_1O = z - 60^\circ.
\end{align*}
\]

Applying the law of sines to triangle \( XO_1O \) establishes the desired result. ■

Now we prove our main result. Without loss of generality, suppose that \( P_4, P_m, \) and \( P_n \) are on sides \( AB, BC, \) and \( CA \), respectively (Figure 5.15). Let \( O \) be the circumcenter of triangle \( ABC \). Without loss of generality, we assume that the circumradius of triangle \( ABC \) is 1, so \( |OA| = |OB| = |OC| = 1 \). Let \( X, Y, \) and \( Z \) be the centers of \( P_4, P_m, \) and \( P_n \), respectively.

Because \( |OB| = |OC| \) and \( |YB| = |YC| \), \( BOCY \) is a kite with \( OY \) as its axis of symmetry. Thus, \( \angle BOY = \frac{\angle BOC}{2} = \angle A \), and \( \angle OYB = 180^\circ / m \). Let
\[ \alpha = \frac{180^\circ}{m}. \] Applying the law of sines to triangle \( OBY \), we obtain

\[ |OY| = \frac{\sin (A + \alpha)}{\sin \alpha}. \]

Likewise, by setting \( \angle ZOC = \frac{180^\circ}{n} = \beta \), we have

\[ |OX| = \frac{\sin(C + 45^\circ)}{\sin 45^\circ} = \sqrt{2} \sin(C + 45^\circ) \quad \text{and} \quad |OZ| = \frac{\sin(B + \beta)}{\sin \beta}. \]

Note that \( O \) is inside triangle \( XYZ \), because it is on the respective perpendicular rays from \( X, Y, \) and \( Z \) toward sides \( AB, BC, \) and \( CA \). Because \( \angle BOY = \angle A \) and \( \angle BOX = \angle C \), we find that \( \angle XOY = \angle C + \angle A \). Likewise, \( \angle YOZ = \angle A + \angle B \) and \( \angle ZOX = \angle B + \angle C \). Applying the Lemma gives

\[ \frac{|OY|}{\sin(B + C - 60^\circ)} = \frac{|OZ|}{\sin(C + A - 60^\circ)} = \frac{|OX|}{\sin(A + B - 60^\circ)}, \]

or

\[ \frac{|OY|}{\sin(A + 60^\circ)} = \frac{|OZ|}{\sin(B + 60^\circ)} = \frac{|OX|}{\sin(C + 60^\circ)}. \]

It follows that

\[ \frac{\sin(A + \alpha) \csc \alpha}{\sin(A + 60^\circ)} = \frac{\sin(B + \beta) \csc \beta}{\sin(B + 60^\circ)} = \frac{\sqrt{2} \sin(C + 45^\circ)}{\sin(C + 60^\circ)}. \]

Because \( y = \cot x \) is decreasing for \( x \) with \( 0^\circ \leq x \leq 180^\circ \), by the addition and subtraction formulas, the function

\[ f(x) = \frac{\sin(x - 15^\circ)}{\sin x} = \cos 15^\circ - \cot x \sin 15^\circ \]

is increasing for \( x \) with \( 0^\circ \leq x \leq 90^\circ \). Consequently, because \( 0^\circ \leq C + 60^\circ \leq 150^\circ \) (\( \angle C \leq 90^\circ \)), it follows that

\[ \frac{\sqrt{2} \sin(C + 45^\circ)}{\sin(C + 60^\circ)} \leq \frac{\sqrt{2} \sin(90^\circ + 45^\circ)}{\sin(90^\circ + 60^\circ)} = 2, \]

with equality if and only if \( C = 90^\circ \). Therefore,

\[ \frac{\sin(A + \alpha) \csc \alpha}{\sin(A + 60^\circ)} = \frac{\sin(B + \beta) \csc \beta}{\sin(B + 60^\circ)} \leq 2. \quad \text{(*)} \]

Because triangle \( ABC \) is nonobtuse, at least two of its angles are between \( 45^\circ \) and \( 90^\circ \). Without loss of generality, we may assume that \( 45^\circ \leq B \leq 90^\circ \),
so \( \sin(B + 60^\circ) > 0 \) and \( \cot B \leq 1 \). Then from the second inequality in the relation \((*)\), we have

\[
\sin(B + \beta) \csc \beta \leq 2 \sin(B + 60^\circ),
\]

or

\[
\sin B \cot \beta + \cos B \leq \sin B + \sqrt{3} \cos B,
\]

by the addition and subtraction formulas. Dividing both sides of the above inequality by \( \sin B \) yields

\[
\cot \beta \leq 1 + (\sqrt{3} - 1) \cot B \leq 1 + \sqrt{3} - 1 = \sqrt{3},
\]

implying that \( \beta \geq 30^\circ \). But since \( n \geq 6 \), \( \beta = 180^\circ / n \leq 30^\circ \). Thus all the equalities hold, and so \( \angle C = 90^\circ \) and \( \angle A = \angle B = 45^\circ \), as claimed.

50. [MOSP 2000] Let \( ABC \) be an acute triangle. Prove that

\[
\left( \frac{\cos A}{\cos B} \right)^2 + \left( \frac{\cos B}{\cos C} \right)^2 + \left( \frac{\cos C}{\cos A} \right)^2 + 8 \cos A \cos B \cos C \geq 4.
\]

**Note:** It is easier to rewrite the above inequality in terms of \( \cos^2 A \), \( \cos^2 B \), and \( \cos^2 C \). By Introductory Problem 24(d), we have

\[
4 - 8 \cos A \cos B \cos C = 4 \left( \cos^2 A + \cos^2 B + \cos^2 C \right).
\]

It suffices to prove

\[
\left( \frac{\cos A}{\cos B} \right)^2 + \left( \frac{\cos B}{\cos C} \right)^2 + \left( \frac{\cos C}{\cos A} \right)^2 \geq 4 \left( \cos^2 A + \cos^2 B + \cos^2 C \right).
\]

We present three approaches.

**First Solution:** By the weighted arithmetic–geometric means inequality, we have

\[
2 \left( \frac{\cos A}{\cos B} \right)^2 + \left( \frac{\cos B}{\cos C} \right)^2 \geq 3 \frac{\cos^4 A}{\frac{\cos^2 B \cos^2 C}{3 \cos^2 A}}
\]

\[
= \frac{\sqrt{\cos^2 A \cos^2 B \cos^2 C}}{3 \cos^2 A}
\]

\[
\geq 12 \cos^2 A,
\]
by Introductory Problem 28(a). Adding the above inequality with its analogous forms and dividing both sides of the resulting inequality by 3, we obtain inequality (†).

**Second Solution:** Setting \( x = \frac{\cos B}{\cos C}, \ y = \frac{\cos C}{\cos A}, \ z = \frac{\cos A}{\cos B} \) in Advanced Problem 42(a) yields

\[
\left( \frac{\cos A}{\cos B} \right)^2 + \left( \frac{\cos B}{\cos C} \right)^2 + \left( \frac{\cos C}{\cos A} \right)^2 = x^2 + y^2 + z^2 \\
\geq 2(\cos A \cos B + \cos B \cos C + \cos C \cos A) \\
= 2 \left[ \frac{\cos C \cos A}{\cos B} + \frac{\cos A \cos B}{\cos C} + \frac{\cos B \cos C}{\cos A} \right].
\]

However, setting

\[
x = \sqrt{\frac{\cos B \cos C}{\cos A}}, \quad y = \sqrt{\frac{\cos A \cos B}{\cos C}}, \quad z = \sqrt{\frac{\cos C \cos A}{\cos B}},
\]
in Advanced Problem 42(a) again, we find that

\[
2 \left[ \frac{\cos C \cos A}{\cos B} + \frac{\cos A \cos B}{\cos C} + \frac{\cos B \cos C}{\cos A} \right] = 2(x^2 + y^2 + z^2) \\
\geq 4(\cos A + \cos B + \cos C) \\
= 4 \left( \cos^2 A + \cos^2 B + \cos^2 C \right),
\]

by noting that

\[
\cos A = \cos A \sqrt{\frac{\cos C \cos B}{\cos C} \cdot \frac{\cos A \cos B}{\cos B}} = \cos^2 A
\]

and its analogous forms for \( \cos B \) and \( \cos C \).

**Third Solution:** The result follows from the following Lemma.

**Lemma** For positive real numbers \( a, b, c \) such that \( abc \leq 1 \),

\[
\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c.
\]

**Proof:** Replacing \( a, b, c \) by \( ta, tb, tc \) with \( t = 1/\sqrt[3]{abc} \) leaves the left-hand side of the inequality unchanged and increases the value of the right-hand
side and results in the equality $a b t c t = a b c t^3 = 1$. Hence we may assume without loss of generality that $a b c = 1$. Then there exist positive real numbers $x, y, z$ such that $a = x/y, b = z/x, c = y/z$. The rearrangement inequality gives

$$x^3 + y^3 + z^3 \geq x^2 z + y^2 x + z^2 y.$$ 

Thus

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = \frac{x^3 + y^3 + z^3}{xyz} \geq \frac{x^2 z + y^2 x + z^2 y}{xyz} = \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = a + b + c,$$

as desired. \[\blacksquare\]

Now we prove our main result. Note that

$$\left(4 \cos^2 A\right) \left(4 \cos^2 B\right) \left(4 \cos^2 C\right) = (8 \cos A \cos B \cos C)^2 \leq 1$$

by Introductory Problem 28(a). Setting $a = 4 \cos^2 A, b = 4 \cos^2 B, c = 4 \cos^2 C$ in the Lemma yields

$$\left(\frac{\cos A}{\cos B}\right)^2 + \left(\frac{\cos B}{\cos C}\right)^2 + \left(\frac{\cos C}{\cos A}\right)^2 = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c = 4(\cos^2 A + \cos^2 B + \cos^2 C),$$

establishing inequality $(\dagger)$.

51. For any real number $x$ and any positive integer $n$, prove that

$$\left| \sum_{k=1}^{n} \frac{\sin k x}{k} \right| \leq 2 \sqrt{\pi}.$$ 

Solution: The solution is based on the following three Lemmas.

**Lemma 1** Let $n$ be a positive integer, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two sequences of real numbers. Then

$$\sum_{k=1}^{n} a_k b_k = S_n b_n + \sum_{k=1}^{n-1} S_k (b_k - b_{k+1}),$$
where \( S_k = a_1 + a_2 + \cdots + a_k \), for \( k = 1, 2, \ldots, n \).

**Proof:** Set \( S_0 = 0 \). Then \( a_k = S_k - S_{k-1} \) for \( k = 1, 2, \ldots, n \), and so

\[
\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (S_k - S_{k-1}) b_k = \sum_{k=1}^{n} S_k b_k - \sum_{k=1}^{n} S_{k-1} b_k \\
= S_n b_n + \sum_{k=1}^{n-1} S_k b_k - \sum_{k=2}^{n} S_{k-1} b_k - S_0 b_1 \\
= S_n b_n + \sum_{k=1}^{n-1} S_k b_k - \sum_{k=1}^{n-1} S_k b_{k+1} \\
= S_n b_n + \sum_{k=1}^{n-1} S_k (b_k - b_{k+1}),
\]

as desired.

---

**Lemma 2**  [Abel’s inequality] Let \( n \) be a positive integer, and let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be two sequences of real numbers with \( b_1 \geq b_2 \geq \cdots \geq b_n \geq 0 \). Then

\[
m b_1 \leq \sum_{k=1}^{n} a_k b_k \leq M b_1,
\]

where \( S_k = a_1 + a_2 + \cdots + a_k \), for \( k = 1, 2, \ldots, n \), and \( M \) and \( m \) are the maximum and minimum, respectively, of \( \{ S_1, S_2, \ldots, S_n \} \).

**Proof:** Note that \( b_n \geq 0 \) and \( b_k - b_{k+1} \geq 0 \) for \( k = 1, 2, \ldots, n-1 \). Lemma 1 gives

\[
\sum_{k=1}^{n} a_k b_k = S_n b_n + \sum_{k=1}^{n-1} S_k (b_k - b_{k+1}) \\
\leq M b_n + M \sum_{k=1}^{n-1} (b_k - b_{k+1}) = M b_1.
\]

establishing the second desired inequality. In exactly the same way, we can prove the first desired inequality.

---

**Lemma 3**  Let \( x \) be a real number that is not an even multiple of \( \pi \), then

\[
\left| \sum_{k=m+1}^{n} \frac{\sin kx}{k} \right| \leq \frac{1}{(m+1) \left| \sin \frac{x}{2} \right|}.
\]
where \( m \) and \( n \) are positive integers with \( m < n \).

**Proof:** For \( k = 1, 2, \ldots, n-m \), let \( a_k = \sin[(k+m)x] \sin \frac{x}{2} \) and \( b_k = \frac{1}{k+m} \).

Then by Lemma 2, we have

\[
\frac{s}{m+1} = sb_1 \leq \sum_{k=m+1}^{n} \frac{\sin kx \sin \frac{x}{2}}{k} = \sum_{k=1}^{n-m} a_kb_k \leq Sb_1 = \frac{S}{m+1},
\]

where \( S_k = a_1 + a_2 + \cdots + a_k \), and \( S = \max\{S_1, S_2, \ldots, S_n\} \) and \( s = \min\{S_1, S_2, \ldots, S_n\} \). The **product-to-sum formulas** give

\[
2a_i = 2 \sin[(i+m)x] \sin \frac{x}{2} = \cos \left(i + m - \frac{1}{2}\right)x - \cos \left(i + m + \frac{1}{2}\right)x,
\]

and so

\[
2S_k = 2a_1 + 2a_2 + \cdots + 2a_k = \cos \left(m + \frac{1}{2}\right)x - \cos \left(k + m + \frac{1}{2}\right)x.
\]

It follows that \(-2 \leq 2S_k \leq 2\) for \( k = 1, 2, \ldots, n \), and so \(-1 \leq s \leq S \leq 1\). Consequently,

\[
-\frac{1}{m+1} \leq -b_1 \leq \sum_{k=m+1}^{n} \frac{\sin kx \sin \frac{x}{2}}{k} \leq Sb_1 \leq b_1 = \frac{1}{m+1},
\]

implying that

\[
\left| \sum_{k=m+1}^{n} \frac{\sin kx \sin \frac{x}{2}}{k} \right| \leq \frac{1}{m+1},
\]

from which the desired result follows. \( \square \)

Now we are ready to prove our main result. Because \( y = |\sin x| \) is a periodic function with period \( \pi \), we may assume that \( x \) is in the interval \((0, \pi)\). (Note that the desired result is trivial for \( x = 0 \).) For a fixed \( x \) with \( 0 < x < \pi \), let \( m \) be the nonnegative integer such that

\[
m \leq \frac{\sqrt{\pi}}{x} < m + 1.
\]

Thus

\[
\left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| \leq \left| \sum_{k=1}^{m} \frac{\sin kx}{k} \right| + \left| \sum_{k=m+1}^{n} \frac{\sin kx}{k} \right|.
\]
5. Solutions to Advanced Problems

Here we set the first summation on the right-hand side to be 0 if $m = 0$, and the first summation taken from 1 to $n$ and the second to be 0 if $m \geq n$. It suffices to show that

$$\left| \sum_{k=1}^{m} \frac{\sin kx}{k} \right| \leq \sqrt{\pi} \tag{*}$$

and

$$\left| \sum_{k=m+1}^{n} \frac{\sin kx}{k} \right| \leq \sqrt{\pi}. \tag{**}$$

Because $|\sin x| < x$ and by the definition of $m$, it follows that

$$\left| \sum_{k=1}^{m} \frac{\sin kx}{k} \right| \leq \sum_{k=1}^{m} \frac{kx}{k} = \sum_{k=1}^{m} x = mx \leq \sqrt{\pi},$$

establishing inequality (*). On the other hand, by Lemma 3, we have

$$\left| \sum_{k=m+1}^{n} \frac{\sin kx}{k} \right| \leq \frac{1}{(m+1)} \left| \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right|.$$

Note that $y = \sin x$ is concave for $0 < x < \frac{\pi}{2}$. Thus, the graph of $y = \sin x$ is above the line connecting the points $(0, 0)$ and $(\frac{\pi}{2}, 1)$ on the interval $(0, \frac{\pi}{2})$; that is, $\sin x > \frac{2x}{\pi}$. Hence for $0 < x < \pi$, we have

$$\sin \frac{x}{2} < \frac{2 \cdot \frac{x}{2}}{\pi} = \frac{x}{\pi}.$$

It follows that

$$\left| \sum_{k=m+1}^{n} \frac{\sin kx}{k} \right| \leq \frac{1}{(m+1)} \left| \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right| \leq \frac{1}{m+1} \cdot \frac{x}{\pi} \leq \frac{\sqrt{\pi}}{x} \cdot \frac{x}{\pi} = \sqrt{\pi},$$

establishing inequality (**). Our proof is thus complete.