3
Advanced Problems

1. Two exercises on $\sin k^\circ \sin (k + 1)^\circ$:

   (a) Find the smallest positive integer $n$ such that
   
   \[
   \frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.
   \]

   (b) Prove that
   
   \[
   \frac{1}{\sin 1^\circ \sin 2^\circ} + \frac{1}{\sin 2^\circ \sin 3^\circ} + \cdots + \frac{1}{\sin 89^\circ \sin 90^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.
   \]

2. Let $ABC$ be a triangle, and let $x$ be a nonnegative real number. Prove that
   
   \[
a^x \cos A + b^x \cos B + c^x \cos C \leq \frac{1}{2} (a^x + b^x + c^x).
   \]

3. Let $x, y, z$ be positive real numbers.
(a) Prove that
\[ \frac{x}{\sqrt{1 + x^2}} + \frac{y}{\sqrt{1 + y^2}} + \frac{z}{\sqrt{1 + z^2}} \leq \frac{3\sqrt{3}}{2} \]
if \( x + y + z = xyz \);

(b) Prove that
\[ \frac{x}{1 - x^2} + \frac{y}{1 - y^2} + \frac{z}{1 - z^2} \geq \frac{3\sqrt{3}}{2} \]
if \( 0 < x, y, z < 1 \) and \( xy + yz + zx = 1 \).

4. Let \( x, y, z \) be real numbers with \( x \geq y \geq z \geq \frac{\pi}{12} \) such that \( x + y + z = \frac{\pi}{2} \).
Find the maximum and the minimum values of the product \( \cos x \sin y \cos z \).

5. Let \( ABC \) be an acute-angled triangle, and for \( n = 1, 2, 3 \), let
\[ x_n = 2^{n-3} (\cos^n A + \cos^n B + \cos^n C) + \cos A \cos B \cos C. \]
Prove that
\[ x_1 + x_2 + x_3 \geq \frac{3}{2}. \]

6. Find the sum of all \( x \) in the interval \([0, 2\pi]\) such that
\[ 3 \cot^2 x + 8 \cot x + 3 = 0. \]

7. Let \( ABC \) be an acute-angled triangle with side lengths \( a, b, c \) and area \( K \).
Prove that
\[ \sqrt{a^2b^2 - 4K^2} + \sqrt{b^2c^2 - 4K^2} + \sqrt{c^2a^2 - 4K^2} = \frac{a^2 + b^2 + c^2}{2}. \]

8. Compute the sums
\[ \binom{n}{1} \sin a + \binom{n}{2} \sin 2a + \cdots + \binom{n}{n} \sin na \]
and
\[ \binom{n}{1} \cos a + \binom{n}{2} \cos 2a + \cdots + \binom{n}{n} \cos na. \]
9. Find the minimum value of
\[ | \sin x + \cos x + \tan x + \sec x + \csc x | \]
for real numbers \( x \).

10. Two real sequences \( x_1, x_2, \ldots \) and \( y_1, y_2, \ldots \) are defined in the following way:
\[ x_1 = y_1 = \sqrt{3}, \quad x_{n+1} = x_n + \sqrt{1 + x_n^2}, \quad y_{n+1} = \frac{y_n}{1 + \sqrt{1 + y_n^2}}. \]
for all \( n \geq 1 \). Prove that \( 2 < x_n y_n < 3 \) for all \( n > 1 \).

11. Let \( a, b, c \) be real numbers such that
\[ \sin a + \sin b + \sin c \geq \frac{3}{2}. \]
Prove that
\[ \sin \left( a - \frac{\pi}{6} \right) + \sin \left( b - \frac{\pi}{6} \right) + \sin \left( c - \frac{\pi}{6} \right) \geq 0. \]

12. Consider any four numbers in the interval \( \left[ \frac{\sqrt{2} - \sqrt{6}}{2}, \frac{\sqrt{2} + \sqrt{6}}{2} \right] \). Prove that there are two of them, say \( a \) and \( b \), such that
\[ \left| a\sqrt{4 - b^2} - b\sqrt{4 - a^2} \right| \leq 2. \]

13. Let \( a \) and \( b \) be real numbers in the interval \( [0, \frac{\pi}{2}] \). Prove that
\[ \sin^6 a + 3 \sin^2 a \cos^2 b + \cos^6 b = 1 \]
if and only if \( a = b \).

14. Let \( x, y, z \) be real numbers with \( 0 < x < y < z < \frac{\pi}{2} \). Prove that
\[ \frac{\pi}{2} + 2 \sin x \cos y + 2 \sin y \cos z \geq \sin 2x + \sin 2y + \sin 2z. \]
15. For a triangle $XYZ$, let $r_{XYZ}$ denote its inradius. Given that the convex pentagon $ABCDE$ is inscribed in a circle, prove that if $r_{ABC} = r_{AED}$ and $r_{ABD} = r_{AEC}$, then triangles $ABC$ and $AED$ are congruent.

16. All the angles in triangle $ABC$ are less then $120^\circ$. Prove that
\[
\frac{\cos A + \cos B - \cos C}{\sin A + \sin B - \sin C} > -\frac{\sqrt{3}}{3}.
\]

17. Let $ABC$ be a triangle such that
\[
\left(\cot \frac{A}{2}\right)^2 + \left(2 \cot \frac{B}{2}\right)^2 + \left(3 \cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,
\]
where $s$ and $r$ denote its semiperimeter and its inradius, respectively. Prove that triangle $ABC$ is similar to a triangle $T$ whose side lengths are all positive integers with no common divisor and determine these integers.

18. Prove that the average of the numbers
\[
2 \sin 2^\circ, \ 4 \sin 4^\circ, \ 6 \sin 6^\circ, \ \ldots, \ 180 \sin 180^\circ
\]
is $\cot 1^\circ$.

19. Prove that in any acute triangle $ABC$,
\[
\cot^3 A + \cot^3 B + \cot^3 C + 6 \cot A \cot B \cot C \geq \cot A + \cot B + \cot C.
\]

20. Let $\{a_n\}$ be the sequence of real numbers defined by $a_1 = t$ and $a_{n+1} = 4a_n(1-a_n)$ for $n \geq 1$. For how many distinct values of $t$ do we have $a_{2004} = 0$?

21. Triangle $ABC$ has the following property: there is an interior point $P$ such that $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$, and $\angle PAC = 40^\circ$. Prove that triangle $ABC$ is isosceles.
22. Let \( a_0 = \sqrt{2} + \sqrt{3} + \sqrt{6} \), and let \( a_{n+1} = \frac{a_n^2 - 5}{2(a_n + 2)} \) for integers \( n > 0 \). Prove that

\[
a_n = \cot \left( \frac{2^{n-3} \pi}{3} \right) - 2
\]

for all \( n \).

23. Let \( n \) be an integer with \( n \geq 2 \). Prove that

\[
\prod_{k=1}^{n} \tan \left( \frac{\pi}{3} \left( 1 + \frac{3^k}{3^n - 1} \right) \right) = \prod_{k=1}^{n} \cot \left( \frac{\pi}{3} \left( 1 - \frac{3^k}{3^n - 1} \right) \right).
\]

24. Let \( P_2(x) = x^2 - 2 \). Find all sequences of polynomials \( \{P_k(x)\}_{k=1}^{\infty} \) such that \( P_k(x) \) is monic (that is, with leading coefficient 1), has degree \( k \), and \( P_i(P_j(x)) = P_j(P_i(x)) \) for all \( i \) and \( j \).

25. In triangle \( ABC \), \( a \leq b \leq c \). As a function of angle \( \angle C \), determine the conditions under which \( a + b - 2R - 2r \) is positive, negative, or zero.

26. Let \( ABC \) be a triangle. Points \( D, E, F \) are on sides \( BC, CA, AB \), respectively, such that \( |DC| + |CE| = |EA| + |AF| = |FB| + |BD| \). Prove that

\[
|DE| + |EF| + |FD| \geq \frac{1}{2}(|AB| + |BC| + |CA|).
\]

27. Let \( a \) and \( b \) be positive real numbers. Prove that

\[
\frac{1}{\sqrt{1 + a^2}} + \frac{1}{\sqrt{1 + b^2}} \geq \frac{2}{\sqrt{1 + ab}}
\]

if either (1) \( 0 < a, b \leq 1 \) or (2) \( ab \geq 3 \).

28. Let \( ABC \) be a nonobtuse triangle such that \( |AB| > |AC| \) and \( \angle B = 45^\circ \). Let \( O \) and \( I \) denote the circumcenter and incenter of triangle \( ABC \), respectively. Suppose that \( \sqrt{2}|OI| = |AB| - |AC| \). Determine all the possible values of \( \sin A \).
29. Let \( n \) be a positive integer. Find the real numbers \( a_0 \) and \( a_{k,\ell} \), \( 1 \leq \ell < k \leq n \), such that
\[
\frac{\sin^2 n x}{\sin^2 x} = a_0 + \sum_{1 \leq \ell < k \leq n} a_{k,\ell} \cos (k - \ell)x
\]
for all real numbers \( x \) not an integer multiple of \( \pi \).

30. Let \( S \) be the set of all triangles \( ABC \) for which
\[
5 \left( \frac{1}{|AP|} + \frac{1}{|BQ|} + \frac{1}{|CR|} \right) - \frac{3}{\min(|AP|, |BQ|, |CR|)} = \frac{6}{r},
\]
where \( r \) is the inradius and \( P \), \( Q \), and \( R \) are the points of tangency of the incircle with sides \( AB \), \( BC \), and \( CA \), respectively. Prove that all triangles in \( S \) are isosceles and similar to one another.

31. Let \( a, b, c \) be real numbers in the interval \( (0, \frac{\pi}{2}) \). Prove that
\[
\frac{\sin a \sin(a - b) \sin(a - c)}{\sin(b + c)} + \frac{\sin b \sin(b - c) \sin(b - a)}{\sin(c + a)} + \frac{\sin c \sin(c - a) \sin(c - b)}{\sin(a + b)} \geq 0.
\]

32. Let \( ABC \) be a triangle. Prove that
\[
\sin \frac{3A}{2} + \sin \frac{3B}{2} + \sin \frac{3C}{2} \leq \cos \frac{A - B}{2} + \cos \frac{B - C}{2} + \cos \frac{C - A}{2}.
\]

33. Let \( x_1, x_2, \ldots, x_n, n \geq 2, \) be \( n \) distinct real numbers in the interval \([-1, 1]\). Prove that
\[
\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} \geq 2^{n-2},
\]
where \( t_i = \prod_{j \neq i} |x_j - x_i| \).

34. Let \( x_1, \ldots, x_{10} \) be real numbers in the interval \([0, \pi/2]\) such that \( \sin^2 x_1 + \sin^2 x_2 + \cdots + \sin^2 x_{10} = 1 \). Prove that
\[
3(\sin x_1 + \cdots + \sin x_{10}) \leq \cos x_1 + \cdots + \cos x_{10}.
\]
35. Let \(x_1, x_2, \ldots, x_n\) be arbitrary real numbers. Prove the inequality
\[
\frac{x_1}{1 + x_1^2} + \frac{x_2}{1 + x_1^2 + x_2^2} + \cdots + \frac{x_n}{1 + x_1^2 + \cdots + x_n^2} < \sqrt{n}.
\]

36. Let \(a_0, a_1, \ldots, a_n\) be numbers in the interval \(\left(0, \frac{\pi}{2}\right)\) such that
\[
\tan \left(\frac{a_0 - \pi}{4}\right) + \tan \left(\frac{a_1 - \pi}{4}\right) + \cdots + \tan \left(\frac{a_n - \pi}{4}\right) \geq n - 1.
\]
Prove that
\[
\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.
\]

37. Find all triples of real numbers \((a, b, c)\) such that \(a^2 - 2b^2 = 1, 2b^2 - 3c^2 = 1,\) and \(ab + bc + ca = 1.\)

38. Let \(n\) be a positive integer, and let \(\theta_i\) be angles with \(0 < \theta_i < 90^\circ\) such that
\[
\cos^2 \theta_1 + \cos^2 \theta_2 + \cdots + \cos^2 \theta_n = 1.
\]
Prove that
\[
\tan \theta_1 + \tan \theta_2 + \cdots + \tan \theta_n \geq (n - 1)(\cot \theta_1 + \cot \theta_2 + \cdots + \cot \theta_n).
\]

39. One of the two inequalities
\[
\sin x \sin x < \cos x \cos x \quad \text{and} \quad \sin x \sin x > \cos x \cos x
\]
is always true for all real numbers \(x\) such that \(0 < x < \frac{\pi}{4}\). Identify that inequality and prove your result.

40. Let \(k\) be a positive integer. Prove that \(\sqrt{k + 1} - \sqrt{k}\) is not the real part of the complex number \(z\) with \(z^n = 1\) for some positive integer \(n.\)

41. Let \(A_1 A_2 A_3\) be an acute-angled triangle. Points \(B_1, B_2, B_3\) are on sides \(A_2 A_3, A_3 A_1, A_1 A_2\), respectively. Prove that
\[
2(b_1 \cos A_1 + b_2 \cos A_2 + b_3 \cos A_3) \geq a_1 \cos A_1 + a_2 \cos A_2 + a_3 \cos A_3,
\]
where \(a_i = |A_{i+1} A_{i+2}|\) and \(b_i = |B_{i+1} B_{i+2}|\), for \(i = 1, 2, 3\) (with indices taken modulo 3; that is, \(x_{i+3} = x_i\)).
42. Let \(ABC\) be a triangle. Let \(x\), \(y\), and \(z\) be real numbers, and let \(n\) be a positive integer. Prove the following four inequalities.

(a) \[x^2 + y^2 + z^2 \geq 2yz \cos A + 2zx \cos B + 2xy \cos C.\]
(b) \[x^2 + y^2 + z^2 \geq 2(-1)^{n+1}(yz \cos nA + zx \cos nB + xy \cos nC).\]
(c) \[yza^2 + zxb^2 + yxc^2 \leq R^2(x + y + z)^2.\]
(d) \[xa^2 + yb^2 + zc^2 \geq 4[ABC]\sqrt{xy + yz + zx}.\]

43. A circle \(\omega\) is inscribed in a quadrilateral \(ABCD\). Let \(I\) be the center of \(\omega\). Suppose that
\[(|AI| + |DI|)^2 + (|BI| + |CI|)^2 = (|AB| + |CD|)^2.\]
Prove that \(ABCD\) is an isosceles trapezoid.

44. Let \(a, b,\) and \(c\) be nonnegative real numbers such that
\[a^2 + b^2 + c^2 + abc = 4.\]
Prove that
\[0 \leq ab + bc + ca - abc \leq 2.\]

45. Let \(s, t, u,\) and \(v\) be numbers in the interval \((0, \frac{\pi}{2})\) with \(s + t + u + v = \pi\). Prove that
\[
\frac{\sqrt{2} \sin s - 1}{\cos s} + \frac{\sqrt{2} \sin t - 1}{\cos t} + \frac{\sqrt{2} \sin u - 1}{\cos u} + \frac{\sqrt{2} \sin v - 1}{\cos v} \geq 0.
\]

46. Suppose a calculator is broken and the only keys that still work are the \(\sin, \cos, \tan, \sin^{-1}, \cos^{-1},\) and \(\tan^{-1}\) buttons. The display initially shows 0. Given any positive rational number \(q\), show that we can get \(q\) to appear on the display panel of the calculator by pressing some finite sequence of buttons. Assume that the calculator does real-number calculations with infinite precision, and that all functions are in terms of radians.
47. Let $n$ be a fixed positive integer. Determine the smallest positive real number $\lambda$ such that for any $\theta_1, \theta_2, \ldots, \theta_n$ in the interval $(0, \frac{\pi}{2})$, if
\[
\tan \theta_1 \tan \theta_2 \cdots \tan \theta_n = 2^{n/2},
\]
then
\[
\cos \theta_1 + \cos \theta_2 + \cdots + \cos \theta_n \leq \lambda.
\]

48. Let $ABC$ be an acute triangle. Prove that
\[
\begin{align*}
&(\sin 2B + \sin 2C)^2 \sin A + (\sin 2C + \sin 2A)^2 \sin B \\
&\quad + (\sin 2A + \sin 2B)^2 \sin C \leq 12 \sin A \sin B \sin C.
\end{align*}
\]

49. On the sides of a nonobtuse triangle $ABC$ are constructed externally a square $P_4$, a regular $m$-sided polygon $P_m$, and a regular $n$-sided polygon $P_n$. The centers of the square and the two polygons form an equilateral triangle. Prove that $m = n = 6$, and find the angles of triangle $ABC$.

50. Let $ABC$ be an acute triangle. Prove that
\[
\left(\frac{\cos A}{\cos B}\right)^2 + \left(\frac{\cos B}{\cos C}\right)^2 + \left(\frac{\cos C}{\cos A}\right)^2 + 8 \cos A \cos B \cos C \geq 4.
\]

51. For any real number $x$ and any positive integer $n$, prove that
\[
\left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| \leq 2\sqrt{\pi}.
\]